

Eugene Paul Wigner's Nobel Prize

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Abstract

In 1963, Eugene Paul Wigner was awarded the Nobel Prize in Physics for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles. There are no disputes about this statement. On the other hand, there still is a question of why the statement did not mention Wigner's 1939 paper on the Lorentz group, which was regarded by Wigner and many others as his most important contribution in physics. By many physicists, this paper was regarded as a mathematical exposition having nothing to do with physics. However, it has been more than one half century since 1963, and it is of interest to see what progress has been made toward understanding physical implications of this paper and its historical role in physics. Wigner in his 1963 paper defined the subgroups of the Lorentz group whose transformations do not change the four-momentum of a given particle, and he called them the little groups. Thus, Wigner's little groups are for internal space-time symmetries of particles in the Lorentz-covariant world. Indeed, this subgroup can explain the electron spin and spins of other massive particles. However, for massless particles, there was a gap between his little group and electromagnetic waves derivable Maxwell's equations. This gap was not completely removed until 1990. The purpose of this report is to review the stormy historical process in which this gap is cleared. It is concluded that Wigner's little groups indeed can be combined into one Lorentz-covariant formula which can dictate the symmetry of the internal space-time symmetries of massive and massless particles in the Lorentz covariant world, just like Einstein's energy-momentum relation applicable to both slow and massless particles.

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1 Introduction

Let us start with Isaac Newton. He formulated his gravity law applicable to two point particles. It took him 20 years to extend his law for solid spheres with non-zero radii.

In 1905, Albert Einstein formulated his special relativity and was interested in how things look to moving observers. He met Niels Bohr occasionally. Bohr was interested in the electron orbit of the hydrogen atom. Then, they could have talked about how the orbit looks to a moving observer [1], as illustrated in Fig. 1. If they did, we do not know anything about it. Indeed, it is for us to settle this Bohr-Einstein issue.

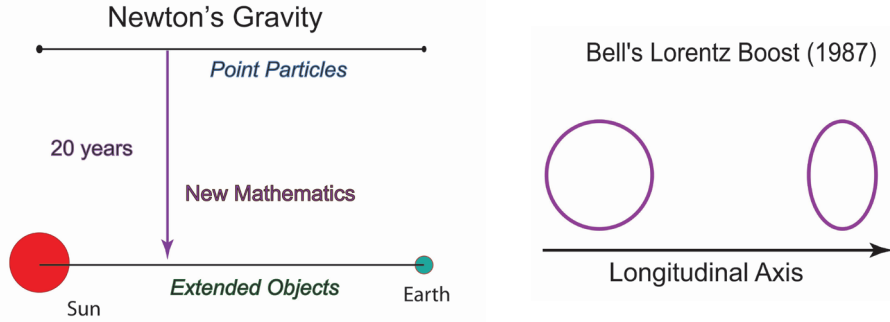


Figure 1: Newton's gravity law for point particles and extended objects. It took him 20 years to formulate the same law for extended objects. As for the classical picture of Lorentz contraction of the electron orbit in the hydrogen atom, it is expected that the longitudinal component becomes contracted while the transverse components are not affected. In the first edition of his book published in 1987, 60 years after 1927, John S. Bell included this picture of the orbit viewed by a moving observer [1]. While talking about quantum mechanics in his book, Bell overlooked the fact that the electron orbit in the hydrogen atom had been replaced by a standing wave in 1927. The question then is how standing waves look to moving observers.

The purpose of the present paper is to discuss whether Wigner's 1939 paper on the Lorentz group provides the framework to address the internal space-time symmetries of particles in the Lorentz-covariant world. This question is far more important than whether Wigner deserved a Nobel prize for this paper alone.

For many years since 1963, many people claimed that Wigner's 1939 paper is worthless because he did not get the Nobel prize for it. Let us respond to this fatalistic view. Einstein did not get the prize for his formulation of special relativity in 1905. Does this mean that Einstein's special relativity worthless? We shall return to this question in the Appendix.

However, it is quite possible that Wigner started subject, but did not finish it. If so, how did this happen? In his 1939 paper [2], Wigner considered the subgroups of

Table 1: One little group for both massive and massless particles. Einstein's special relativity gives one relation for both. Wigner's little group unifies the internal space-time symmetries for massive and massless particles which are locally isomorphic to $O(3)$ and $E(2)$ respectively. This table is from Ref. [5].

	Massive, Slow	COVARIANCE	Massless, Fast
Energy-Momentum	$E = p^2/2m$	Einstein's $E = \sqrt{p^2 + m^2}$	$E = p$
Internal space-time symmetry	S_3 S_1, S_2	Wigner's Little Group	S_3 Gauge Transformations

the Lorentz group whose transformations leave the four-momentum of a given particle invariant. These subgroups are called Wigner's little groups and dictate the internal space-time symmetries in the Lorentz-covariant world.

He observed first that a massive particle at rest has three rotational degree of freedom leading to the concept of spin. Thus the little group for this massive particle is like $O(3)$. How about this massive particle moving in the z direction. We could settle this issue easily.

Wigner observed also that a massless particle cannot be brought to its rest frame, but he showed that the little group for the massless particle also has three degrees of freedom, and that this little is locally isomorphic to the group $E(2)$ or the two-dimensional Euclidean group. This means that generators of this little group share the same set of closed commutation relations with that for two-dimensional Euclidean group with one rotational and two translational degrees of freedom.

It is not difficult to associate the rotational degree of freedom of $E(2)$ to the helicity of the massless particle. However, what is the physics of the those two translational degrees of freedom? Wigner did not provide the answer to this question in his 1939 paper [2]. Indeed, this question has a stormy history, and the issue was not completely settled until 1990 [3], fifty one years after 1939, or 27 years after his Nobel prize in 1963.

For many years, the major complaint had been that his little groups could not explain the Maxwell field. Is it possible to construct electromagnetic four-potential and the

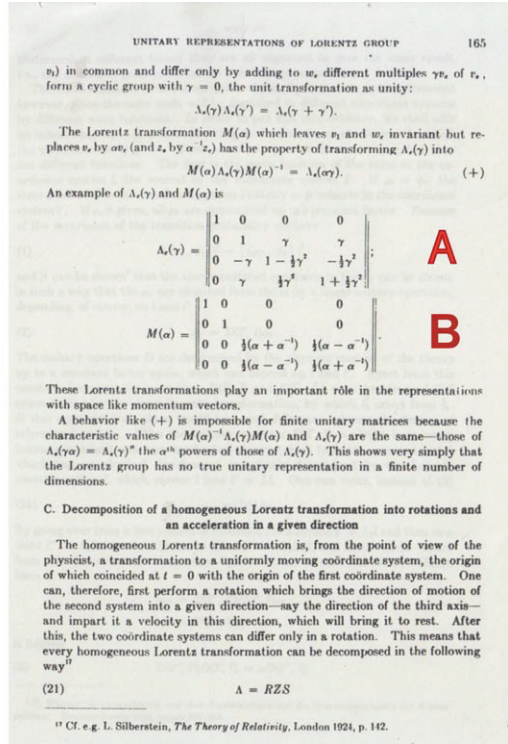
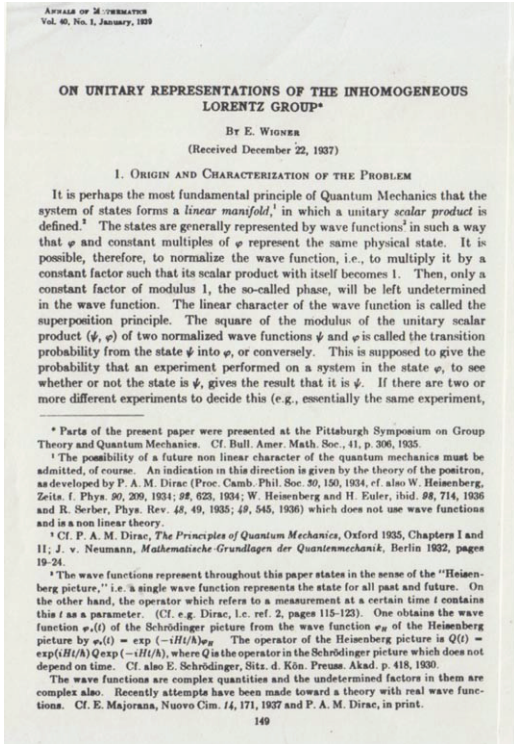


Figure 2: Wigner's 1939 paper in the Annals of Mathematics. Its front page is on page 149 of the journal. On page 165, there are two strange matrices. The second matrix (matrix B) is for the Lorentz boost along the z direction. However, the physics of the first matrix (matrix A) was not completely understood until 1990, 51 years after 1939.

Maxwell tensor as representations of representations of Wigner's little group for massless particles?

To answer this question, let us go to one of his matrices in his paper, given in Fig. 2. It is easy to see that matrix B is for a Lorentz boost along the z direction. Matrix A leaves the four-momentum of the massless particle invariant. What else does this matrix do? In 1972 [4], Kuperzstych showed that it performs a gauge transformation when applied to the electromagnetic four-potential, but he did not see this as Wigner's problem. Indeed, this question was not completely answered until 1990 [3].

In the present paper, we point out the complete understanding of this matrix leads to result given in Table 1 contained the paper I published with my younger colleagues in 1986 [5]. As Einstein's energy-momentum leads to its expressions both in the small-momentum and large-momentum limits, Wigner's little groups explain the internal space-time symmetries for the massive particle at rest as well as for the massless particle, as summarized in Table 1.

From Sec. 2 to Sec. 5, technical details are given. The present author gave many lectures on this subject in the past. In this report, he explains the same subject at a public-lecture level by avoiding group theoretical words as much as possible. Since this paper deals with a sensitive issue, it is appropriate to mention his background and as well as his experience in dealing with those people who did not agree with him.

In Sec. 2, Wigner's little groups are spelled out in the language of four-by-four matrices. In Sec. 3, the two-by-two representation is given for spin-half particles. The gauge transformation is defined for this two-by-two representation. In Sec. 4, it is shown that the gauge transformation defined in the two-by-two representation leads to the gauge transformation applicable to the four-potential for photons. In Sec. 5, the $O(3)$ -like little group for massive particles.

In the Appendix, it is noted that this paper deals with a very serious historical issue. The question then is whether the present author is qualified to write about this issue. Thus, it is appropriate to explain how and why he had to study Wigner's paper. A more interesting story is how much resistance he had to face in making his results known to Wigner and to the world.

2 Wigner's little groups

If we use the four-vector convention $x^\mu = (x, y, z, t)$, the generators of rotations around and boosts along the z axis take the form

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (1)$$

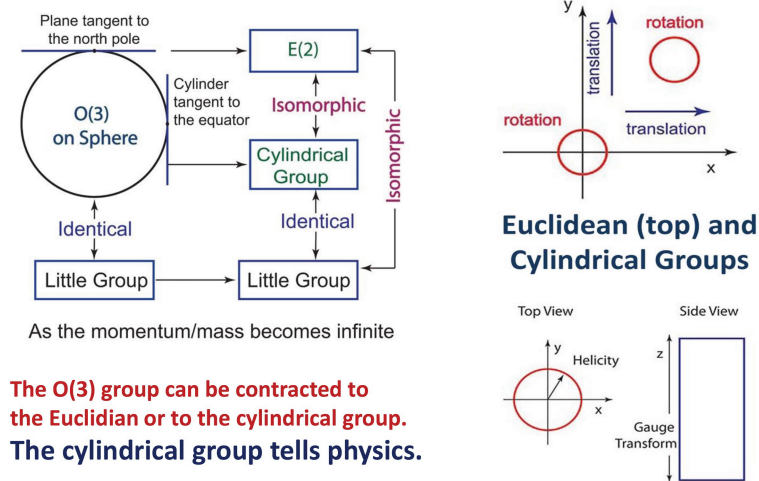


Figure 3: Transformations of the $E(2)$ group and the cylindrical group. They share the same Lie algebra, but only the cylindrical group leads to a geometrical interpretation of the gauge transformation.

respectively. We can also write the four-by-four matrices for J_1 and J_2 for the rotations around the x and y directions, as well as K_1 and K_2 for Lorentz boosts along the x and y directions respectively [6, 7]. These six generators satisfy the following set of commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (2)$$

This closed set of commutation relations is called the Lie algebra of the Lorentz group. The three J_i operators constitute a closed subset of this Lie algebra. Thus, the rotation group is a subgroup of the Lorentz group.

In addition, Wigner in 1939 considered a subgroup generated by [2]

$$J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (3)$$

These generators satisfy the closed set of commutation relations

$$[N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \quad (4)$$

As Wigner observed in 1939 [2], this set of commutation relations is just like that for the generators of the two-dimensional Euclidean group with one rotation and two translation generators, as illustrated in Fig. 3. However, the question is what aspect of the massless particle can be explained in terms of this two-dimensional geometry.

Indeed, this question has a stormy history, and was not answered until 1987. In their paper of 1987 [8], Kim and Wigner considered the surface of a circular cylinder as shown

in Fig. 3. For this cylinder, rotations are possible around the z axis. It is also possible to make translations along the z axis as shown in Fig. 3. We can write these generators as

$$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad (5)$$

applicable to the three-dimensional space of (x, y, z) . They then satisfy the closed set of commutation relations

$$[Q_1, Q_2] = 0, \quad [L_3, Q_1] = iQ_2, \quad [L_3, Q_2] = -iQ_1. \quad (6)$$

which becomes that of Eq.(4) when Q_1, Q_2 , and L_3 are replaced by N_1, N_2 , and J_3 of Eq.(3) respectively. Indeed, this cylindrical group is locally isomorphic to Wigner's little group for massless particles.

Let us go back to the generators of Eq.(3). The role of J_3 is well known. It generates rotations around the momentum and corresponds to the helicity of the massless particle. The N_1 and N_2 matrices take the form [6, 7]

$$N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \quad (7)$$

The transformation matrix is

$$D(u, v) = \exp \{-i(uN_1 + vN_2)\} = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}. \quad (8)$$

In his 1939 paper [2], Wigner observed that this matrix leaves the four-momentum of the massless particle invariant as can be seen from

$$\begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_3 \end{pmatrix}, \quad (9)$$

but he never attempted to apply this matrix to the photon four-potential.

It is interesting to note that in 1976 noted that this form is applicable to the four-potential while making rotation and boosts whose combined effects do not change the four-momentum [4] of the photon. In 1981, Han and Kim carried out the same calculation within the framework of Wigner's little group [9]. Kuperzstych's conclusion was that the four-by-four matrix of Eq.(8) performs a gauge transformation when applied

to the photon four-potential, and Han and Kim arrived at the same conclusion. Let us see how this happens.

Let us next consider the electromagnetic wave propagating along the z direction:

$$A^\mu(z, t) = (A_1, A_2, A_3, A_0)e^{i\omega(z-t)}, \quad (10)$$

and apply the $D(U, v)$ matrix to this electromagnetic four-vector:

$$\begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix}, \quad (11)$$

which becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} - (A_3 - A_0) \begin{pmatrix} u \\ v \\ (u^2 + v^2)/2 \\ (u^2 + v^2)/2 \end{pmatrix}. \quad (12)$$

If the four-vector satisfies the Lorentz condition $A_3 = A_0$, this expression becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ uA_1 + vA_3 \\ uA_1 + vA_3 \end{pmatrix}. \quad (13)$$

The net effect is an addition of the same quantity to the longitudinal and time-like components while leaving the transverse components invariant. Indeed, this is a gauge transformation.

3 Spin-1/2 particles

Let us go back to the Lie algebra of the Lorentz group given in Eq.(2). It was noted that there are six four-by-four matrices satisfying nine commutation relations. It is possible to construct the same Lie algebra with six two-by-two matrices [6, 7]. They are

$$J_i = \frac{1}{2}\sigma_i, \quad \text{and} \quad K_i = \frac{i}{2}\sigma_i, \quad (14)$$

where σ_i are the Pauli spin matrices. While J_i are Hermitian, K_i are not. They are anti-Hermitian. Since the Lie algebra of Eq.(2) is Hermitian invariant, we can construct the same Lie algebra with

$$J_i = \frac{1}{2}\sigma_i, \quad \text{and} \quad \dot{K}_i = -\frac{i}{2}\sigma_i. \quad (15)$$

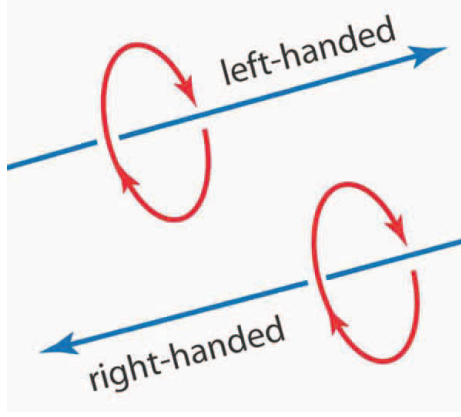


Figure 4: Polarization of massless neutrinos. Massless neutrinos are left-handed, while anti-neutrinos are right-handed. This is a consequence of gauge invariance.

This is the reason why the four-by-four Dirac matrices can explain both the spin-1/2 particle and the anti-particle.

Thus the most general form of the transformation matrix takes the form

$$T = \exp \left(-\frac{i}{2} \sum_i \theta_i \sigma_i + \frac{1}{2} \sum_i \eta_i \sigma_i \right), \quad (16)$$

and this transformation matrix is applicable to the spinors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (17)$$

In addition, we have to consider the transformation matrices

$$\dot{T} = \exp \left(-\frac{i}{2} \sum_i \theta_i \sigma_i - \frac{1}{2} \sum_i \eta_i \sigma_i \right), \quad (18)$$

applicable to

$$\dot{\chi}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \dot{\chi}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (19)$$

With this understanding, let us go back to the Lie algebra of Eq.(2). Here again the rotation generators satisfy the closed set of commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [\dot{J}_i, \dot{J}_j] = i\epsilon_{ijk} \dot{J}_k. \quad (20)$$

These operators generate the rotation-like $SU(2)$ group, whose physical interpretation is well known, namely the electron and positron spins.

Here also we can consider the $E(2)$ -like subgroup generated by

$$J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (21)$$

The N_1 and N_2 matrices take the form

$$N_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (22)$$

On the other hand, in the ‘‘dotted’’ representation,

$$\dot{N}_1 = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad \dot{N}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (23)$$

There are therefore two different D matrices:

$$D(u, v) = \exp\{- (iuN_1 + ivN_2)\} = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \quad (24)$$

and

$$\dot{D}(u, v) = \exp\{- (iu\dot{N}_1 + iv\dot{N}_2)\} = \begin{pmatrix} 1 & 0 \\ u + iv & 1 \end{pmatrix}. \quad (25)$$

These are the gauge transformation matrices applicable to massless spin-1/2 particles [7, 10].

Here are talking about the Dirac equation for with four-component spinors.

The spinors χ_+ and $\dot{\chi}_-$ are gauge-invariant since

$$D(u, v)\chi_+ = \chi_+, \quad \text{and} \quad \dot{D}(u, v)\dot{\chi}_- = \dot{\chi}_-. \quad (26)$$

As for χ_- and $\dot{\chi}_+$,

$$\begin{aligned} D(u, v)\chi_- &= \chi_- + (u - iv)\chi_+, \\ \dot{D}(u, v)\dot{\chi}_+ &= \dot{\chi}_+ + (u + iv)\dot{\chi}_-. \end{aligned} \quad (27)$$

They are not invariant under the D transformations, and they are not gauge-invariant. Thus, we can conclude that the polarization of massless neutrinos is a consequence of gauge invariance, as illustrated in Fig. 4.

4 Four-vectors from the spinors

We are familiar with the way in which the spin-1 vector is constructed from the spinors in non-relativistic world. We are now interested in constructing four-vectors from these spinors. First of all, with four of the spinors given above, we can start with the products.

$$\chi_i\chi_j, \quad \chi_i\dot{\chi}_j, \quad \dot{\chi}_i\chi_j, \quad \dot{\chi}_i\dot{\chi}_i. \quad (28)$$

resulting in spin-0 scalars and four-vectors and four-by-four tensors for the spin-1 states [7]. The four-vector can be constructed from the combinations $\chi_i\dot{\chi}_j$ and $\dot{\chi}_i\chi_j$.

Among them, let us consider the combinations, let us consider the four resulting from $\dot{\chi}_i \chi_j$. Among them, As far as the rotation subgroup is concerned, $\dot{\chi}_+ \chi_+$, and $\dot{\chi}_- \chi_-$ are like $-(x + iy)$ and $(x - iy)$ respectively, and invariant under Lorentz boosts along the z direction. In addition, we should consider

$$\frac{1}{2}(\dot{\chi}_- \chi_+ + \dot{\chi}_+ \chi_-), \quad \text{and} \quad \frac{1}{2}(\dot{\chi}_- \chi_+ - \dot{\chi}_+ \chi_-), \quad (29)$$

which are invariant under rotations around the z axis. When the system boosted along the z direction, these combinations are transformed like z and t directions respectively.

With these aspects in mind, let us consider the matrix

$$M = \begin{pmatrix} \dot{\chi}_- \chi_+ & \dot{\chi}_- \chi_- \\ -\dot{\chi}_+ \chi_+ & -\dot{\chi}_+ \chi_- \end{pmatrix}, \quad (30)$$

and write the transformation matrix T of Eq.(16) as

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{with} \quad \det(T) = 1. \quad (31)$$

If four matrix elements are complex numbers, there are eight independent parameters. However, the condition $\det(T) = 1$ reduces this number to six. The Lorentz group starts with six degrees of freedom.

It is then possible to write the four-vector (x, y, z, t) as

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}, \quad (32)$$

with its Lorentz-transformation property

$$X' = T X T^\dagger, \quad (33)$$

The four-momentum can also be written as

$$P = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}, \quad (34)$$

with the the transformation property same as that for X given in Eq.(33).

With this understanding, we can write the photon four-potential as

$$A = \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix} \quad (35)$$

Let us go back the two-by-two matrices $D(u, v)$ and $\dot{D}(u, v)$ given in Eqs.(24) and (25). We said there that they perform gauge transformations on massless neutrinos. It is indeed gratifying to note that they also lead to the gauge transformation applicable to the photon four-potential.

$$D(u, v)AD^\dagger(u, v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u + iv & 1 \end{pmatrix}. \quad (36)$$

This results in

$$\begin{pmatrix} A_0 + A_3 + 2(uA_1 + vA_2) & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix} + (A_0 - A_3) \begin{pmatrix} u^2 + v^2 & u - iv \\ u + iv & 1 \end{pmatrix}. \quad (37)$$

If we apply the Lorentz condition $A_0 = A_3$, this matrix becomes

$$\begin{pmatrix} 2A_0 + 2(uA_1 + vA_2) & A_1 - iA_2 \\ A_1 + iA_2 & 0 \end{pmatrix}. \quad (38)$$

This result is the same as the gauge transformation in the four-by-four representation given in Eq.(13).

5 Massless particle as a limiting case of a massive particle

In his 1939 paper [2], Wigner discussed his little groups for massive and massless particles as two distinct mathematical devices. Indeed, Inönü and Wigner in 1953 initiated of the unification of these little groups by observing considering a flat plane tangent to a sphere, while the plane and sphere correspond to the $E(2)$ and $O(3)$ symmetries respectively [11]. This unification was completed in 1990 [3]. The issue is whether the $E(2)$ -like little group can be obtained as a zero-mass limit of the $O(3)$ -like little group for massive particles. Another version of this limiting process is given in Sec. 5 of the present report.

As for the internal space-time symmetry of particles, let us go back to Bohr and Einstein. Bohr was interested in the electron orbit of the hydrogen atom while Einstein was worrying about how things look to moving observers. They met occasionally before and after 1927 to discuss physics. Did they talk about how the stationary hydrogen atom would look to a moving observer? If they did, we do not know about it.

This problem is not unlike the case of Newton's law of gravity. Newton worked out the inverse square law for two point particles. It took him 20 years to work out the same law for extended objects such as the sun and earth, as illustrated in Fig. 1.

In 1905, Einstein formulated his special relativity for point particles. It is for us to settle the issue of how the electron orbit of the hydrogen atom looks to moving observers. Indeed, the circle and ellipse as given in Fig. 1 have been used to illustrate this relativistic effect. However, these figures do not take into account the fact that the electron orbit had been replaced by a standing wave. Indeed, we should learn how to Lorentz-boost standing waves.

Yes, we know how to construct standing waves for the hydrogen atom. Do we know how to Lorentz-boost this atom? The answer is No. However, we can replace it with the proton without changing quantum mechanics. Both the hydrogen atom and the proton are quantum bound states, but the proton can be accelerated. While the Coulomb force is applicable to the hydrogen, the harmonic oscillator potential is used as the simplest binding force for the quark model [12]. We can switch the Coulomb wave functions with oscillator wave functions without changing quantum mechanics. This problem is illustrated in Fig. 9. Then it is possible to construct the oscillator wave functions as a representation of Wigner's little group [6, 7, 13]. In this two-by-two representation, the Lorentz boost along the positive direction is

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad (39)$$

the rotation around the y axis is

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \quad (40)$$

Then, the boosted rotation matrix is

$$B(\eta)R(\theta)B(-\eta) = \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \quad (41)$$

If η becomes very large, and this matrix is to remain finite, θ has to become very small, and this expression becomes [14]

$$\begin{pmatrix} 1 - r^2 e^{-2\eta}/2 & r \\ -r e^{-2\eta} & 1 - r^2 e^{-2\eta}/2 \end{pmatrix}. \quad (42)$$

with

$$r = -\frac{1}{2}\theta e^\eta. \quad (43)$$

This expression becomes

$$D(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}. \quad (44)$$

In this two-by-two representation, the rotation around the z axis is

$$Z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \quad (45)$$

respectively. Thus

$$D(u, v) = Z(\phi)D(r)Z^{-1}(\phi), \quad (46)$$

which becomes

$$D(u, v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \quad (47)$$

with

$$u = r \cos \phi, \quad \text{and} \quad v = r \sin \phi, \quad (48)$$

Here, we have studied how the little group for the $O(3)$ -like little group the massive particle becomes the $E(2)$ -like little group for the massless particle in the infinite- η limit. What does this limit mean physically? The parameter η can be derived from the speed of of the particle. We know $\tanh(\eta) = v/c$, where v is the speed of the particle. Then

$$\tanh \eta = \frac{p}{\sqrt{m^2 + p^2}}, \quad (49)$$

where m and p are the mass and the momentum of the particle respectively. If m is much smaller than p ,

$$e^\eta = \frac{\sqrt{2}p}{m}, \quad (50)$$

which becomes large when m becomes very small. Thus, the limit of large η means the zero-mass limit.

Let us carry our the same limiting process for the four-by-four representation. From the generators of the Lorentz group, it is possible to construct the four-by-four matrices for rotations around the y axis and Lorentz boosts along the z axis as [7]

$$R(\theta) = \exp(-i\theta J_2), \quad \text{and} \quad B(\eta) = \exp(-i\eta K_3), \quad (51)$$

respectively. The Lorentz-boosted rotation matrix is $B(\eta)R(\theta)B(-\eta)$ which can be written as

$$\begin{pmatrix} \cos \theta & 0 & (\sin \theta) \cosh \eta & -(\sin \theta) \sinh \eta \\ 0 & 1 & 0 & 0 \\ -(\sin \theta) \cosh \eta & 0 & \cos \theta - (1 - \cos \theta) \sinh^2 \eta & (1 - \cos \theta) (\cosh \eta) \sinh \eta \\ -(\sin \theta) \cosh \eta & 0 & -(1 - \cos \theta) (\cosh \eta) \sinh \eta & \cos \theta + (1 - \cos \theta) \cosh^2 \eta \end{pmatrix}. \quad (52)$$

While $\tanh \eta = v/c$, this boosted rotation matrix becomes a transformation matrix for a massless particle when η becomes infinite. On the other hand, if the matrix is to be finite in this limit, the angle θ has to become small. If we let $r = -\frac{1}{2}\theta e^\eta$ as given in Eq.(43), this four-by-four matrix becomes

$$\begin{pmatrix} 1 & 0 & -r & r \\ 0 & 1 & 0 & 0 \\ r & 0 & 1 - r^2/2 & r^2/2 \\ r & 0 & -r^2/2 & 1 + r^2/2 \end{pmatrix}. \quad (53)$$

This is the Lorentz-boosted rotation matrix around the y axis. However, we can rotate this y axis around the z axis by ϕ . Then the matrix becomes

$$\begin{pmatrix} 1 & 0 & -r \cos \phi & r \cos \phi \\ 0 & 1 & -r \sin \phi & r \sin \phi \\ r \cos \phi & r \sin \phi & 1 - r^2/2 & r^2/2 \\ r \cos \alpha & r \sin \phi & -r^2/2 & 1 + r^2/2 \end{pmatrix}. \quad (54)$$



**Wightman (middle) and
Wigner's daughter (left) (2002)**



Michel with Wigner (1984)

Figure 5: Arthur Wightman and Louis Michel. Wightman's photo was taken at a luncheon meeting held at Princeton University to celebrate Wigner's 100-th birth day in 2002. Michel's photo was taken during the 13th International Colloquium on Group Theoretical Methods in Physics held at the University of Maryland in 1984. Both photos were taken by the author of this paper.

This matrix becomes $D(u, v)$ of Eq.(8), if replace $r \cos \phi$ and $r \sin \phi$ with u and v respectively, as given in Eq.(48).

A Author's Qualifications

In this report, I am dealing with a very serious issue in physics. The question is whether I am qualified to talk about Wigner's 1939 paper. The reader of this article is not likely to be the first one to raise this issue.

Louis Michel and Arthur Wightman were among the most respected physicists on the Lorentz group, and their photos are in Fig. 5. In 1961, while I was a graduate student at Princeton, I was in Wightman's class. I learned from him the ABC of the Lorentz group. Wightman gave the same set of lectures in France, and he published an article in French [15].

In 1962, Louis Michel gave a series of very comprehensive lectures at a conference held in Istanbul. Indeed, I learned from his lecture note [16] how the inhomogeneous Lorentz group is different from the homogenous Lorentz group.

Both Michel and Wightman became upset when I was meeting Wigner during the period from 1985 to 1990. Wightman sent me a letter telling me my papers on Wigner's 1939 paper are wrong. In particular, he said the table given in Table 1 is wrong. I assume he told the same story to Wigner because his office and Wigner's office were in



Toll, Mrs. Toll, Wigner, Kim (1986)



Faculty photo of 1963

Figure 6: Toll, Mrs. Toll, Wigner, and Kim at the Chancellor's Mansion of UMD (1986). The physics faculty photo of UMD (1963). Kim is the youngest man standing in the middle of the second row.

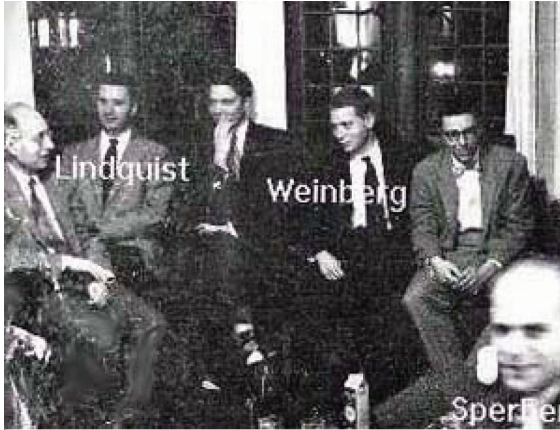
the same building on the campus of Princeton University.

Louis Michel became very angry when I had to tell him I could carry out my Wigner program without his permission. He told me he did not like what I say in Table 1. He even wrote a letter to John S. Toll telling him to reduce my position at the University of Maryland in 1987. Toll was the chancellor of the state-wide University of Maryland system at that time.

He was John A. Wheeler's student at Princeton University and came to the University of Maryland in 1953 to build the present form of the physics department. In 1962, he hired me as an assistant professor one year after my PhD degree at Princeton. Toll became very happy whenever Wigner came to Maryland at my invitation, as indicated in Fig. 6.

In spite of those hostile reactions from Michel and Wightman, Wigner liked Table 1, and continued listening to me. John S. Toll continued supporting my position at the University of Maryland. In spite of what I said above I still like Michel and Wightman. They were great teachers to me.

Stephen Adler and Gerald Feinberg were also very influential physicists during the period from 1960 to 1990. I knew them well. In 1981, when I submitted a paper with my younger colleagues on the Wigner issue to the Physical Review Letters, Feinberg wrote a referee report saying Wigner's 1939 paper is a useless mathematical exposition having nothing to do with physics. He was so proud of what he was saying that he revealed his name in his report, while the referees are anonymous. Thus, he deserves to



Weinberg talking to Wigner (1957)



Biedenharn with Wigner (1988)

Figure 7: Steven Weinberg and Lawrence Biedenharn. They had their own positive views toward Wigner’s 1939 paper. In this photo, Weinberg is talking to Wigner in 1957, when he was a graduate student at Princeton. Biedenharn is standing with Wigner in 1988 during the first Wigner Symposium held at the University of Maryland.

be mentioned in the present report.

Since Feinberg did not give other reasons in his report, we resubmitted the paper asking the editor to examine its scientific contents. At that time, Adler was in the editorial position to make the final decision. Adler said he agreed with Feinberg without making any comments of his own. In other words, Adler was also saying that Wigner’s 1939 paper is worthless. In effect, both Adler and Feinberg were telling us not to waste my time because Wigner did not get the Nobel prize for this paper.

Steven Weinberg was different. In 1964, he published a series of papers on the spin states that can be constructed from Wigner’s little groups [17, 18, 19]. Indeed, he realized that Wigner’s little groups are for the internal space-time symmetries. As for massless particles, Weinberg realized the matrix A of Fig. 2 was troublesome, and constructed his “state vectors” which are independent of this matrix [18, 19]. Does Weinberg’s result bring Wigner’s paper closer to the Maxwell theory?

In the Maxwell formalism, it is possible to construct gauge-independent states, namely electromagnetic fields. It is also possible to construct the electromagnetic four-potential which depends on gauge transformations. Thus, it is not difficult to guess Weinberg’s state vectors are for the electromagnetic field, while matrix A of Fig. 2 is for gauge transformations applicable to the four-potential [7].

With this point in mind, I published in 1981 a paper with Han saying that Matrix A performs a gauge transformation [9]. We considered a momentum-preserving transformation by considering one rotation followed by two boosts, as shown in Fig. 8. We

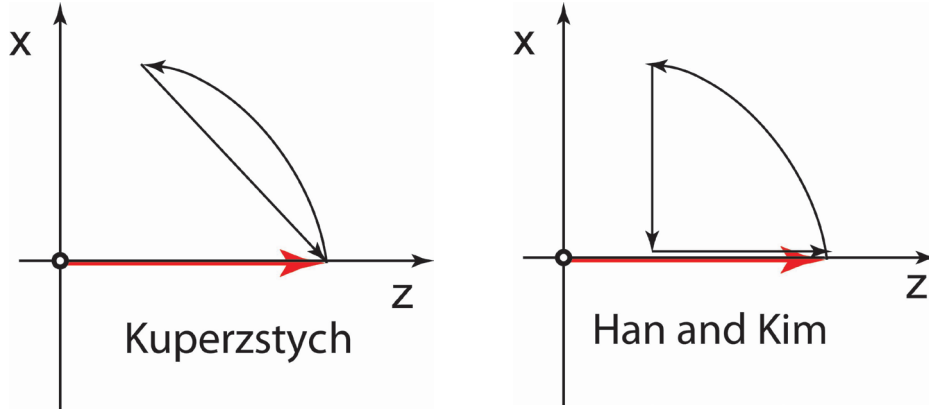


Figure 8: Lorentz transformations which leave the four-momentum invariant. However, these transformations do not lead to identity matrices. When applied to the electromagnetic four-potential, they perform gauge transformations. Furthermore, since two-successive Lorentz boosts lead to one boost preceded by a rotation (called Wigner rotation) in the kinematics of Han and Kim, their transformation is the same as that of Kuperzstych.

submitted this paper to the American Journal of Physics instead of the Physical Review, because we felt that we were not the first ones to observe this. Indeed, in 1972, Kuperzstych got essentially the same result, as indicated also in Fig. 8. It is remarkable that he got this result without making reference to Wigner’s 1939 paper. He concluded his paper saying that the concept of spin could be generated from his momentum-preserving transformation. It is remarkable that he derived this result without relying on the concept of the little groups spelled out in Wigner’s 1939 paper [2].

In 1953, Inönü and Wigner published a paper on group contractions [11]. We can study the contraction of the $O(3)$ group to $E(2)$ by considering a sphere for $O(3)$ and a two-dimensional plane for $E(2)$. We can then consider this plane tangent to the sphere at the north-pole. If the radius of the sphere becomes very large, the spherical surface at the north-pole becomes flat enough to accommodate the $E(2)$ symmetry. We can construct a flat football field on the surface of the earth. Thus, the $E(2)$ -like little group for massless particles can be obtained from the $O(3)$ -like little group for massive particles. Then, what is the physics of the large-radius limit?

In his 1939 paper [2], Wigner considered the little group of the massive particle at rest. What is then the little group for a particle moving along the z direction. The answer is very simple. It is a Lorentz-boosted rotation matrix. What happens when the momentum becomes infinite?

In order to address this question, we can start from Einstein’s $E = \sqrt{m^2 + p^2}$. We all know the form of this relation in the limit of small p/m . We also know the form for

the large- p/m limit. With this point in mind, with Han and Son, I published a paper telling that the the rotational degrees of freedom around x and y directions become one gauge degree of freedom while the rotation around the z axis remains as the helicity degree of freedom, as p/m becomes infinite [20]. After several stages of refinements, we published Table 1 in the Journal of Mathematical Physics in 1986 [5].

Wigner liked this table. This is precisely the reason why I was able to publish a number of papers with him. However, Wigner pointed out to me that the geometry of the two-dimensional plane cannot explain the gauge transformation, as indicated in Fig. 3. We thus worked hard to figure out the solution to this problem. For a given sphere, we can consider also a cylinder tangential to the equatorial belt, as well as a plane tangential to the north pole, as illustrated in Fig. 3. We published this result in the Journal of Mathematical Physics in 1987 [8], and another paper in 1990 [3]. Lawrence Biedenharn was the chief editor of the Journal. He was very happy to publish these papers and congratulated me on reactivating Wigner's 1939 paper.

I am very happy to include in this report Biedenharn's photo with Wigner in Fig. 7, which I took with my Canon AE camera in 1988. Included in the same figure is a photo of Weinberg talking to Wigner while he was a graduate student at Princeton in 1957. Dieter Brill contributed this photo. Weinberg was Sam Treiman's first PhD student and got his degree in 1957. Since I went to Princeton in 1958, I did not have overlapping years with Weinberg, but I had to read his thesis to copy his style of writing. I still like his English style.

Of course, I am proud of working with Wigner during his late years. On the other and, could I do this job without my own background? I had to fix up Wigner's work in order to put my own physics program on a solid theoretical ground. When I was graduate student, and for several years after my PhD degree, I lived in the world where the origin of physics is believed to be in the complex plane of the S-matrix, and bound states should be described by poles in the complex plane.

In 1965, when I pointed out those poles do not necessarily lead to localized wave functions [23], I had to face a stiff resistance from the influential members of the American physics community. I choose not to mention their names. They told me wave functions have nothing to do with physics. This is a totally destructive attitude. However, I took their reactions constructively. We do not know how to Lorentz-boost bound-state wave functions, while the S-matrix is a product of the Lorentz-covariant field theory.

Thus, my problem was to find at least one wave function that can be Lorentz-boosted. I then realized that Dirac in 1945 [24] and Yukawa in 1953 [25] constructed a Gaussian form that can be Lorentz-boosted.

In April of 1970, at the spring meeting of the American physical Society, Feynman gave a talk where he repeatedly mentioned wave functions. He was talking about hadrons as bound states of quarks. My colleagues were saying Feynman was absolutely crazy, but he was a savior to me. Let us face the difficulty of boosting wave functions. In 1971 [12],

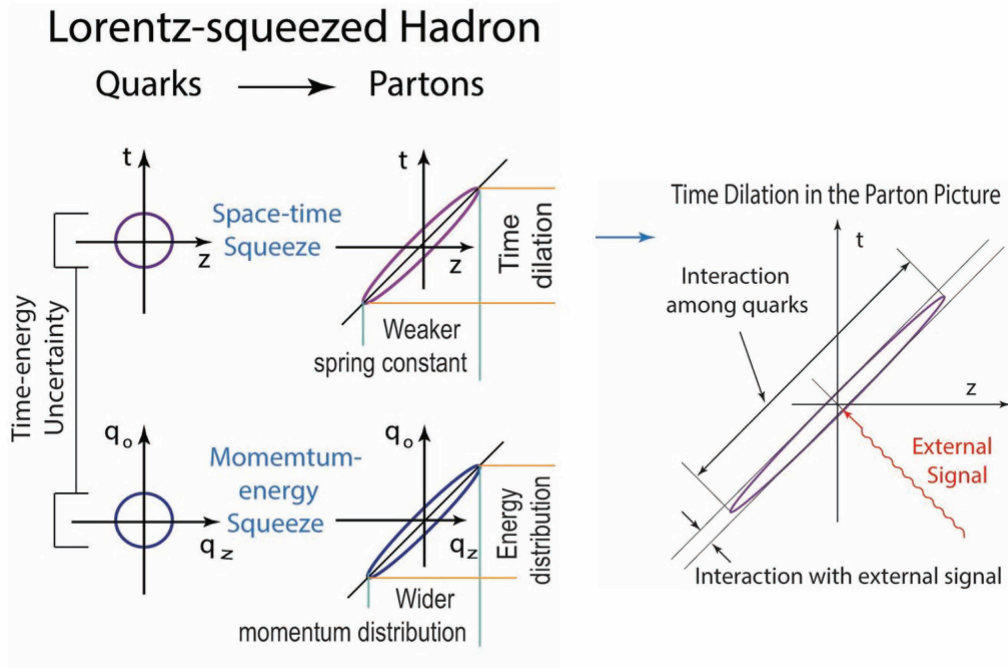


Figure 9: Gell-Mann's Quark model and Feynman's parton model as one Lorentz-covariant entity. The circle-ellipse diagram is from Ref. [6], and also from Refs. [21]. This Lorentz-squeeze is known as the space-time entanglement in the current literature [22].

Table 2: One little group for both massive and massless particles. In this table, we have added the last row to Table 1 telling Gell-Mann's quark model and Feynman's parton model are two different manifestations of one Lorentz-covariant entity. This table is from Ref. [6] and also from Ref. [21].

	Massive, Slow	COVARIANCE	Massless, Fast
Energy-Momentum	$E = p^2/2m$	Einstein's $E = \sqrt{p^2 + m^2}$	$E = p$
Internal space-time symmetry	S_3 S_1, S_2	Wigner's Little Group	S_3 Gauge Transformations
Moving H-atom	Gell-Mann's Quark Model	Covariant Bound State	Feynman's Parton Model

with his students, Feynman published a paper based on his 1970 talk [12]. There, they start with a Lorentz-invariant differential equation for particles bound together by harmonic-oscillator potential. However, they produce solutions containing the Gaussian form

$$\exp \left\{ -\frac{1}{2} (x^2 + y^2 + z^2 - t^2) \right\}. \quad (55)$$

This form is invariant under Lorentz transformations, but it is not possible to physical interpretations to t variable.

On the other hand, Feynman's differential equation also produces the solutions containing the Gaussian form

$$\exp \left\{ -\frac{1}{2} (x^2 + y^2 + z^2 + t^2) \right\}. \quad (56)$$

In this case, the wave function is normalizable in all the variables. The form is not invariant. This means the wave function appears differently to moving observers. Figure 9 illustrates how differently the wave function look differently.

With Marilyn Noz, I used the Gaussian form of Eq.(56) to show that Gell-Mann's quark model and Feynman's parton model are two different limits of one Lorentz-covariant entity, and submitted our result to Physical Review Letters. The referee was very competent and honest. He/she said he/she really learned what the parton model is all about and the result is important. However, he would "enthusiastically" recommend publication in the Comments and Addenda section of the Phys. Rev. D, instead of PRL. We accepted his/her recommendation and published paper as a comment [26].

However, what aspect of the fundamental symmetry does this quark-parton reflect? In order to answer this question, I had to study Wigner's 1939 paper, and show that the Lorentz-covariant oscillator wave functions are representations of Wigner's $O(3)$ -like little group [6, 13]. My continued efforts led to a PRL paper of 1989 [21]. In that paper, I expanded Table 1 to Table 2. This paper also contains the major portion of Fig. 9. The elliptic squeeze described in this figure is called the space-time entanglement in the current literature [22].

Let me summarize what I said above. Many people told me I am totally isolated from the rest of the physics world while working on the problem nobody else worries about. I disagree. I have been in touch with all essential physicists in this field, including Eugene Paul Wigner. In other words, I am qualified to write this report.

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