Photons, neutrinos, and gauge transformations

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(Received 24 May 1985; accepted for publication 9 October 1985)

A unified description is presented for internal space-time symmetries of photons and neutrinos. Lorentz-boosted Dirac equation for a massive spin-1/2 particle is discussed in the Weyl representation. In the large-momentum/small-mass limit, the resulting large component of the Dirac spinor becomes invariant under transformations of the E(2)-like little group, and the small component is not invariant under the E(2)-like transformation. The gauge dependence of the photon four-potential can thus be derived from a direct product of the gauge-independent large component and the gauge-dependent small component.

I. INTRODUCTION

On massless particles, there are a number of important questions for which answers are not readily available. When we discuss blackbody radiation, we assume that the photon spin can be either parallel or antiparallel to the momentum. Why can photons, while being spin-1 particles, not have three different polarizations? While photons can have two different directions of polarization, why do massless neutrinos have only one direction of polarization?

There is yet another question which bothers us constantly. For a stationary particle with nonzero mass with spin 1, the $l = 1$ spherical harmonics describes its spin orientations. For this nonrelativistic case, we know how to construct the spin-1 states from two Pauli spinors. Why is it not possible to construct a photon state as a direct product of two neutrino states?

The answers to these questions can ultimately be found in Wigner's 1939 paper[1] on the Poincaré group. Wigner's representation theory is based on his little groups. The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of a given particle invariant.[1,2] The little groups for massive and massless particles are like the three-dimensional rotation group [or $O(3)$] and the two-dimensional Euclidean group [or E(2)]. We use the word "like" to indicate that two different groups have the same algebraic property even though they may have different matrix representations.

For photons, it is possible to explain the spin orientation and the gauge dependence in terms of the E(2)-like little group for massless particles.[3–5] However, for massless particles with spin 1/2, we never talk about gauge degrees of freedom. Do such degrees of freedom exist for spin-1/2 particles?

We studied this problem in Ref. 3 using the SL(2,c) spinors and showed that neutrino polarization is due to the requirement of gauge invariance. The purpose of the present paper is to translate the result of Ref. 3 into the language of the Dirac equation. In particular, we are interested in the following questions.

(1) The polarization of neutrinos is usually stated as $\gamma_5 = \pm 1$, and the nature seems to favor the $\gamma_5 = -1$ neutrinos. Does this mean that the $\gamma_5 = +1$ neutrinos are gauge dependent?

(2) Is it possible to form a gauge-dependent four-potential as a direct product of two spinors? If so, where is the source of the gauge dependence in the spinors?

The answers to questions (1) and (2) are "no" and "yes," respectively. The gauge dependence comes from the component of the Dirac spinor which becomes vanishingly small when we take the infinite-momentum/zero-mass limit of the Dirac wave function for a massive particle.

In Sec. II we formulate the problem of internal space-time symmetries of massless particles in terms of the E(2)-like little group applicable to photons.[1–5] As for neutrinos, the Dirac equation in the Weyl representation is presented in Sec. III. It is shown that the two-component solutions of the Dirac equation are gauge invariant. However, if we allow a small nonzero mass, there are two additional components which are noninvariant under transformations of the E(2)-like little group. In Sec. IV, we construct a gauge-dependent four-potential from a direct product of two spinors. It is shown that the above-mentioned E(2)-dependent spinors are responsible for the gauge dependence of the four-potential.

II. E(2)-LIKE LITTLE GROUP FOR MASSLESS PARTICLES

Let us begin with the group of Lorentz transformations. This group is generated by three rotation generators J_i and three boost generators K_i. They satisfy the commutation relations:

$$[J_i,J_j] = i\epsilon_{ijk}J_k, \quad [J_i,K_j] = i\epsilon_{ijk}K_k, \quad [K_i,K_j] = -i\epsilon_{ijk}J_k . \tag{1}$$

For a massive particle in its rest frame, the little group is generated by J_1, J_2, and J_3.[1,2] For a massless particle moving along the z direction, the little group is generated by[1–5]

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \quad \text{and } J_3, \tag{2}$$

satisfying commutation relations:

$$[J_3,N_1] = iN_2, \quad [J_3,N_2] = -iN_1, \quad [N_1,N_2] = 0, \tag{3}$$

which are identical to those for the two-dimensional Euclidean group,[1–4] consisting of rotations around the origin generated by J_3 and translations generated by N_1 and N_2 in a two-dimensional Euclidean space.

As for the above-mentioned massless particle, J_3 is the helicity operator. In order to understand the physical implications of the N operators, let us study in detail the case of photons. For a single free photon moving along the z direction, we can write the photon wave function as

$$A^{\mu}(x) = A^{\mu} e^{i\omega(z-t)} , \tag{4}$$

with the four-vector convention:

$$A^{\mu} = (A_1, A_2, A_3, A_0) . \tag{5}$$

The momentum four-vector is clearly

$$p^{\mu} = (0, 0, \omega, \omega) . \tag{6}$$

Then, as we discussed in our earlier papers,[3,5] the little group applicable to the photon four-potential is generated by

$$J_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}.$$

These matrices satrisfy the commutation relations of Eq. (3). J_3 in this case is the helicity operator, and its physics is well known. The N operators generate the translationlike transformations:

$$D(u,v) = D(u,0) D(0,v) , \tag{8}$$

where

$$D(u,0) = D_1(u) = \exp[iuN_1] ,$$
$$D(0,v) = D_2(v) = \exp[-ivN_2] .$$

We can expand the above formulas in power series,

$$D_1(u) = 1 - iuN_1 - (uN_1)^2/2 , \tag{9}$$
$$D_2(v) = 1 - ivN_2 - (vN_2)^2/2 ,$$

with

$$(N_1)^3 = (N_2)^3 = N_1 (N_2)^2 = N_2 (N_1)^2 = 0 .$$

Therefore, the four-by-four matrices for $D_1(v)$ are quadratic in u and v, respectively. These expression is rather cumbersome.[1,3–5] However, if we use the Lorentz condition:

$$\frac{\partial}{\partial x^{\mu}} [A^{\mu}(x)] = p^{\mu} A_{\mu}(x) = 0 , \tag{10}$$

resulting in $A_3 = A_0$, all the elements in the third and fourth columns of the N matrices of Eq. (7) vanish. The D matrix of Eq. (8) then takes a relatively simple form[3]:

$$D(u,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ v & v & 1 & 0 \\ u & v & 0 & 1 \end{bmatrix}. \tag{11}$$

If this D matrix is applied to the photon four-potential given in Eq. (4), it generates a gauge transformation.[3,5] Specifically, if it is applied to the polarization vectors:

$$\epsilon_{\pm} = (1, \ \pm i, \ 0, \ 0) , \tag{12}$$

which are eigenstates of J_3, the effect is

$$D(u,v)\epsilon_{\pm} = (1, \ \pm i, \ u \pm iv, \ u \pm iv) , \tag{13}$$

which is indeed a gauge transformation.[3,5]

The purpose of the present paper is to see whether we can carry out the same procedure for massless particles with spin 1/2, and exploit its consequences within the framework of the Dirac equation.

III. E(2)-LIKE SYMMETRY OF THE DIRAC EQUATION IN THE WEYL REPRESENTATION

We are familiar with the Dirac equation and its free-particle solutions in the standard representation of the Dirac matrices. On the other hand, the Dirac equation in the Weyl representation is frequently mentioned in the literature. What is the advantage of using the Weyl representation?

The Dirac matrices in the Weyl representation take the form[6]:

$$\gamma = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}, \quad \gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{14}$$

The basic advantage of the Weyl representation is that both the generators of rotations and boosts take the block diagonal form.

$$S_i = \begin{bmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & \frac{1}{2}\sigma_i \end{bmatrix}, \quad K_i = \begin{bmatrix} \frac{i}{2}\sigma_i & 0 \\ 0 & -\frac{i}{2}\sigma_i \end{bmatrix}. \tag{15}$$

The upper and lower components of the four-component Dirac spinors can be separated by the sign of the γ_5 matrix, and therefore by the sign of the boost generators. Indeed, the multiplication of the four-component spinor by $(1 \pm \gamma_5)/2$ is the projection of the upper and lower components, respectively. The resulting two-component spinors are called the chiral or Weyl spinors. We shall call them the Weyl spinors in this paper.

We have no difficulty in recognizing S_i of Eq. (15) as the generators of rotations. In order to check whether K_i are the boost generators, let us start with a massive Dirac particle at rest:

$$U(0) = \begin{bmatrix} \alpha \\ \pm \dot{\alpha} \end{bmatrix}, \quad V(0) = \begin{bmatrix} \pm \beta \\ \dot{\beta} \end{bmatrix}, \tag{16}$$

where the $+$ and $-$ signs specify positive and negative energy states, respectively. These spinors are in the eigenstates of S_3. For the upper component, whose boosts are generated by $K_i = (i/2)\sigma_i$, we use the usual Pauli notation α and β for positive and negative spins along the z direction, respectively. For the lower component whose boosts are generated by $K_i = -(i/2)\sigma_i$, we use the dotted spinors $\dot{\alpha}$ and $\dot{\beta}$. If we boost the spinors of Eq. (16) along the z axis by applying the boost operator $\exp(-i\xi K_3)$,

$$U(\mathbf{p}) = \begin{bmatrix} [\exp(+\xi/2)]\alpha \\ \pm [\exp(-\xi/2)]\dot{\alpha} \end{bmatrix},$$

$$V(\mathbf{p}) = \begin{bmatrix} \pm [\exp(-\xi/2)]\beta \\ [\exp(+\xi/2)]\dot{\beta} \end{bmatrix}. \tag{17}$$

where

$$\xi = \tanh^{-1}(p_z/p_0) .$$

The above spinors constitute the solutions of the Dirac equation for a particle with momentum p_z along the z direction in the Weyl representation. Indeed, K_i of Eq. (15) are the generators of boosts.

With this preparation, we construct the generators of the

E(2)-like little group N_1 and N_2 for the Dirac spinors using the formulas given in Eq. (2). For the upper component, $K_i = (i/2)\sigma_i$, and

$$N_1^{(+)} = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \quad N_2^{(+)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{18}$$

The generators $N_1^{(-)}$ and $N_2^{(-)}$ applicable to the lower component are the Hermitian conjugates of the above expressions. The D transformation matrix applicable the Dirac spinor is

$$D(u,v) = \begin{bmatrix} D^{(+)}(u,v) & 0 \\ 0 & D^{(-)}(u,v) \end{bmatrix}, \tag{19}$$

where

$$\begin{aligned} D^{(+)}(u,v) &= \exp[-i(uN_1^{(+)} + vN_2^{(+)})] \\ &= \begin{bmatrix} 1 & u-iv \\ 0 & 1 \end{bmatrix}, \\ D^{(-)}(u,v) &= \exp[-i(uN_1^{(-)} + vN_2^{(-)})] \\ &= \begin{bmatrix} 1 & 0 \\ -u-iv & 1 \end{bmatrix}. \end{aligned} \tag{20}$$

Since the D matrix in the case of photons performs a gauge transformation, it is appropriate to call the above matrices gauge transformation matrices applicable to Dirac particles.

It is important to note that the gauge transformation property of the upper component is different from that of the lower component. Let us now separate them using the projection operators $(1 \pm \gamma_5)/2$. The resulting Weyl spinors are gauge invariant in the sense that

$$D^{(+)}(u,v)\alpha = \alpha, \quad D^{(-)}(u,v)\dot\beta = \dot\beta. \tag{21}$$

On the other hand, the Weyl spinors are gauge dependent in the sense that

$$\begin{aligned} D^{(+)}(u,v)\beta &= \beta + (u-iv)\alpha, \\ D^{(-)}(u,v)\dot\alpha &= \dot\alpha - (u+iv)\dot\beta. \end{aligned} \tag{22}$$

This result is summarized Table I. The gauge-invariant spinors of Eq. (16) appear as polarized neutrinos in the real world.[3] However, where do the above gauge-dependent spinors stand in the physics of spin-1/2 particles? Why do we not talk about the gauge-dependent spinors?

In order to address these questions, let us go back to the Dirac spinors in Eq. (17). As the momentum/mass becomes large, $\exp(\xi/2)$ becomes large and $\exp(-\xi/2)$ becomes small. From Eqs. (21) and (22), we can see that the large components are gauge invariant while the small components are gauge dependent. Therefore, in general, spin-1/2 particles with nonzero mass are not invariant under gauge transformations.

In the large-momentum/zero-mass limit, it is possible to let[7]

$$\exp(-\xi) = [(p_0 - p_z)/(p_0 + p_z)]^{1/2} \to 0, \tag{23}$$

and the Dirac equation becomes a pair of the Weyl equations with $\gamma_5 = \pm 1$ whose solutions are

$$U(\mathbf{p})\begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad V(\mathbf{p}) = \begin{bmatrix} 0 \\ \dot\beta \end{bmatrix}. \tag{24}$$

These Dirac spinors become invariant under the D transformation, representing polarized neutrinos.[3] Indeed, the gauge-dependent components disappear in the large-momentum/zero-mass limit. This is precisely why we do not talk about gauge transformations on massless neutrinos.

On the other hand, neutrinos with small mass are of current interest.[8] Even if they are not found experimentally in the near future, the concept of small-mass neutrinos will play an important role for many years to come.[9] The small components in the Dirac spinor can no longer be ignored. In Sec. IV, we shall use them to construct gauge-dependent four-potentials.

Table I. The effect of the D transformation on the Weyl spinors. $D^{(+)}$ and $D^{(-)}$ are applicable to undotted and dotted spinors, respectively. α and $\dot\beta$ are invariant under the D transformation, while $\dot\alpha$ and β are not.

	α	β
$D^{(+)}$	$D^{(+)}(u,v)\alpha = \alpha$	$D^{(+)}(u,v)\beta = \beta + (u-iv)\alpha$
$D^{(-)}$	$D^{(-)}(u,v)\dot\alpha = \dot\alpha - (u+iv)\dot\beta$	$D^{(-)}\dot\beta = \dot\beta$

IV. FOUR-VECTORS AND *SL*(2,*c*) SPINORS

As is discussed in the literature,[2,10] it is possible to construct four-vectors from $SL(2,c)$ spinors for massive particles. We use the same procedure to secure gauge dependence of four-potentials from the Weyl spinors discussed in Sec. III. Let us consider the following spinor combinations:

$$\alpha\dot\alpha, \quad (\alpha\dot\beta + \beta\dot\alpha)/\sqrt{2}, \quad \beta\dot\beta. \tag{25}$$

Then, since the rotation operator is the same for both dotted and undotted spinors as is given in Eq. (15), the above spinor combinations should behave like $Y_1^{(1)}(\theta,\phi)$, $Y_1^0(\theta,\phi)$, and $Y_1^{-1}(\theta,\phi)$, respectively, under rotation. The spinor combination

$$(\alpha\dot\beta - \beta\dot\alpha)/\sqrt{2}, \tag{26}$$

remains invariant, and therefore behaves like t under rotation. This takes care of rotations.

As for boosts, we can rewrite the above combinations as

$$\begin{aligned} -\alpha\dot\alpha &= (1,i,0,0), \quad \beta\dot\beta = (1,-i,0,0), \\ \alpha\dot\beta &= (0,0,1,1), \quad \beta\dot\alpha = (0,0,1,-1), \end{aligned} \tag{27}$$

where, as in Eq. (5), we are using the four-vector convention:

$$x^\mu = (x,y,z,t). \tag{28}$$

We drop for simplicity the $\sqrt{2}$ factor for normalization in Eq. (27). Under the boost along the z direction, $\alpha\dot\alpha$ and $\beta\dot\beta$ remain invariant, and $\frac{1}{2}(\alpha\dot\beta \pm \beta\dot\alpha)$ behave like z and t, respectively. As for the x direction, the boost operator is

$$B_x(\eta) = B_x^{(+)}(\eta)\, B_x^{(-)}(\eta), \tag{29}$$

where $B_x^{(+)}(\eta)$ and $B_x^{(-)}(\eta)$ are the boost operators applicable to the undotted and dotted spinors, respectively. They take the form

$$B_x^{(\pm)}(\eta) = \begin{bmatrix} \cosh(\eta/2) & \pm\sinh(\eta/2) \\ \pm\sinh(\eta/2) & \cosh(\eta/2) \end{bmatrix}, \tag{30}$$

respectively. The effect of this boost on $\alpha\dot\alpha$ is

$$\begin{aligned} B_x(\eta)(-\alpha\dot\alpha) &= -\alpha\dot\alpha\cosh^2(\eta/2) + \beta\dot\beta\sinh^2(\eta/2) \\ &\quad + \tfrac{1}{2}(\alpha\dot\beta - \beta\dot\alpha)\sinh\eta \\ &= (\cosh\eta, i, 0, \sinh\eta). \end{aligned} \tag{31}$$

This result is identical to that of boosting $(x+iy)$. We can carry out similar calculations for $B_x(\eta)\beta\dot{\beta}$, and $B_x(\eta)\alpha\dot{\beta}$, and $B_x(\eta)\beta\dot{\alpha}$. As for the boost along the y direction, we know how to rotate this system of spinors around the z axis by 90°.

We have shown above how to construct four-vectors from the Weyl spinors. We are then led to the question of whether the $D(u,v)$ transformation applicable to the spinors leads to a gauge transformation on the four-potential. For $D(u,v)$ to be consistent with Eq. (27), we should choose

$$D(u,v)=D^{(+)}(u,v)D^{(-)}(u,v)\,, \qquad (32)$$

where $D^{(+)}$ and $D^{(-)}$ are applicable to the first and second spinors of Eq. (27), respectively. Then

$$\begin{aligned} D(u,v)(-\alpha\dot{\alpha}) &= -\alpha\dot{\alpha}+(u+iv)\alpha\dot{\beta}\,,\\ D(u,v)\beta\dot{\beta} &= \beta\dot{\beta}+(u-iv)\alpha\dot{\beta}\,,\\ D(u,v)\alpha\dot{\beta} &= \alpha\dot{\beta}\,. \end{aligned} \qquad (33)$$

The spinor combination $\beta\dot{\alpha}$ given in Eq. (27) does not satisfy the Lorentz condition, and should therefore be excluded. The first two equations of the above expression correspond to the gauge transformations on the photon polarization vectors given in Eq. (13). The third equation corresponds to the effect of the D transformation on the four-momentum, confirming the fact that $D(u,v)$ is an element of the little group.[1] Indeed, the D matrices given Eq. (20) are the smallest gauge transformation matrices from which the gauge transformation on the four-potential is derivable.

V. CONCLUDING REMARKS

The Dirac equation in the Weyl representation is mentioned frequently in textbooks. The difference between the standard Dirac representation and the Weyl representation is that γ_0 and γ_5 are interchanged. These two representations are unitarily equivalent. We have shown in this paper that the Weyl representation is very convenient when we discuss Lorentz-transformation properties of spinors, because the transformation matrices are of the block diagonal form.

Using the Dirac equation in the Weyl representation, we studied the E(2)-like little group for massless particles with spin 1/2. It was noted that the translationlike transformations of the E(2) group correspond to gauge transformations. It was noted also that there are gauge-dependent spinors in addition to those which are gauge invariant. It was shown that the massless two-component Dirac spinors contain only gauge-invariant spinors.

It was shown also that, if neutrinos have a small mass, the Dirac spinor should contain four components including those which are not invariant under gauge transformations. Since a four-vector can be constructed as a direct product of two spinors, it is possible to express the usual gauge transformation on the four-potential in terms of the gauge transformation applicable to the gauge-dependent spinors.

ACKNOWLEDGMENTS

We would like to thank S. Nussinov, M. Parida, and K. Tsokos for helpful discussions. We are indebted to O. W. Greenberg for making available to us his unpublished note on the two-neutrino theory of photons.[11]

[1]E. P. Wigner, Ann. Math. **149**, 40 (1939).

[2]V. Bargman and E. P. Wigner, Proc. Natl. Acad. Sci. (US) **34**, 211 (1946).

[3]D. Han, Y. S. Kim, and D. Son, Phys. Rev. D **26**, 3717 (1982).

[4]S. Weinberg, Phys. Rev. **135**, B1049 (1964); *Ibid.* **134**, B882 (1964).

[5]D. Han and Y. S. Kim, Am. J. Phys. **49**, 348 (1981); D. Han, Y. S. Kim, and D. Son, Phys. Rev. D **31**, 328 (1985).

[6]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[7]For the transition of the $O(3)$-like little group to the E(2)-like little group in this limit, see E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. (US) **39**, 510 (1953); S. P. Misra and J. Maharana, Phys. Rev. D **14**, 133 (1976); D. Han, Y. S. Kim, and D. Son, Phys. Lett. B **131**, 327 (1983); D. Han, Y. S. Kim, M. E. Noz, and D. Son, Am. J. Phys. **52**, 1037 (1984).

[8]S. M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 226 (1978); M. Fukugita and T. Yamagida, Phys. Lett. **144B**, 386 (1984).

[9]J. S. Rigden, Am. J. Phys. **49**, 303 (1981).

[10]V. I. Smirnov, *Linear Algebra and Group Theory* (McGraw-Hill, New York, 1961); F. R. Halpern, *Relativistic Quantum Mechanics* (Prentice-Hall, Englewood Cliff, NJ, 1968); Yu. V. Novozhilov, *Introduction to Elementary Particle Theory* (Pergamon, Oxford, 1975).

[11]O. W. Greenberg and A. S. Wightman, Bull. Am. Phys. Soc. **30**, 75 (1955).

A modulation transfer function analyzer based on a microcomputer and dynamic ram chip camera

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(Received 7 August 1985; accepted for publication 17 October 1985)

The modulation transfer function of a lens is measured by adapting a commercially available microcomputer-based detector for use in an optics laboratory. The technique and instrumentation for such measurements are described.

As the field of optics grows and its techniques become more sophisticated, optics education must keep pace with this growth. One of the most important places for training a student to be an effective member in the optics community is the undergraduate optics laboratory. Any current technique that can be introduced into the laboratory that approximates current practice will benefit students and make their studies more relevant. One standard means of specify-