Homework #2 — Phys625 — Spring 2004
Deadline: Wednesday, April 7, 2004.
Turn in homework in the class or put it in the box on the door of Phys 2314 by 3 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the periods, like (3.2.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

Fermi gas

1. [**4 points**] *Green's function in a complete basis*

Consider noninteracting particles characterized by a Hamiltonian $\hat{H}$, which has a complete set of one-particle energy eigenfunctions $\psi_n(\mathbf{r})$ with the eigenvalues ε_n:

$$\hat{H}\psi_n(\mathbf{r}) = \varepsilon_n \psi_n(\mathbf{r}). \tag{1}$$

Starting from the definition of Green's function (4.2.15) and the expansion $\hat{\psi}(\mathbf{r},t) = \sum_n \psi_n(\mathbf{r}) e^{-i(\varepsilon_n-\mu)t}\hat{a}_n$, perform the Fourier transform in time (similarly to Sec. 4.6) and derive the following expression for Green's function:

$$G(\omega,\mathbf{r}_1,\mathbf{r}_2) = \sum_n \frac{\psi_n(\mathbf{r}_1)\,\psi_n^*(\mathbf{r}_2)}{\omega-\varepsilon_n+\mu+i0\,\mathrm{sgn}(\varepsilon_n-\mu)}, \tag{2}$$

where μ is the chemical potential.

Check that function (2) satisfies the equation $(\omega-\hat{H}+\mu)\,G(\omega,\mathbf{r}_1,\mathbf{r}_2) = \delta(\mathbf{r}_1-\mathbf{r}_2)$. Thus, $G = (\omega-\hat{H}+\mu+i0\,\mathrm{sgn}\,\omega)^{-1}$.

2. *Density oscillations in one dimension (1D)*

Consider a one-dimensional (1D) electron gas occupying semi-infinite space $x > 0$ with impenetrable boundary at $x = 0$. The energy dispersion is $\varepsilon(p) = p^2/2m$.

(a) [**2 points**] Using Eq. (2), write down Green's function $G(\omega, x_1, x_2)$. To satisfy the vanishing boundary condition at $x = 0$, use the basis functions $\sin(px)$ in Eq. (2).

(b) [**6 points**] Using the formula

$$n(x) = -2i\int G(\omega,x,x)\,e^{-i0\,\omega}\,\frac{d\omega}{2\pi}, \tag{3}$$

calculate electron density $n(x)$. The factor 2 in Eq. (3) comes from spin. Compare the period of oscillation in x (the so-called Friedel oscillations) with the average distance between electrons.

Hint: Integrate Eq. (3) over ω first, using contour integration in the complex plane of ω. Then conver summation over n into intgration over p and take integral over p.

3. **[6 points]** *Electron-hole continuum*

Let us introduce the operator $\hat{c}^{+}_{\mathbf{p},\mathbf{q}}$ that creates a hole with momentum $\mathbf{p}$ and an electron with momentum $\mathbf{p}$+$\mathbf{q}$ out of the Fermi sea:

$$\hat{c}^{+}_{\mathbf{p},\mathbf{q}} = \hat{a}^{+}_{\mathbf{p}+\mathbf{q}}\hat{a}_{\mathbf{p}}, \quad \text{where} \quad |\mathbf{p}| < p_F \quad \text{and} \quad |\mathbf{p}+\mathbf{q}| > p_F. \tag{4}$$

Show that the energy of such an excitation is

$$E_{\mathbf{p},\mathbf{q}} = \epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{p}} = \frac{|\mathbf{p}+\mathbf{q}|^2 - |\mathbf{p}|^2}{2m}. \tag{5}$$

Argue that $\mathbf{q}$ is the total momentum of the electron-hole pair. Outline the area covered by the electron-hole excitations on a plot of $E(q)$ vs. q for all permitted values of p with the restriction (4). This is the so-called continuum of electron-hole excitations.

Make separate plots for different dimensions of space: $D = 3$, 2, and 1. Hint: You should reproduce Fig. 6.8, but with explicit expressions for the boundaries of the continuum. Watch out for differences between different D!