**Homework #8** — Phys625 — Spring 2002
**Deadline: Thursday, April 18, 2002.**
Turn in homework in the class or put it in the box on the door of Phys 2314 by 10 a.m.

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**Do not forget to write your name and the homework number!**
Equation numbers with the period, like (3.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

# Finite Temperature (Ch. IV)

This homework explores the Matsubara diagram technique for a finite temperature $T$. From operational point of view, it is the same as the diagram technique at zero temperature with the replacement $\omega \to i\omega_n$, where $\omega_n$ is the Matsubara frequency: $\omega_n = \pi T(2n+1)$ for fermions and $\omega_n = 2n\pi T$ for bosons ($n$ is an integer).

**1.** [**6 points**] *Friedel oscillations in 1D* (See Problem 3 of HW 3)

Consider 1D electron gas occupying semi-infinite space $x > 0$ with impenetrable boundary at $x = 0$. The energy dispersion is $\varepsilon(p) = p^2/2m$, but you may use the linearized electron dispersion law near $\pm p_F$.

Calculate electron density $n(x)$ using Eq. (37.15),

$$n(x) = 2T \sum_n \mathcal{G}(i\omega_n, x, x)\, e^{i0\,\omega_n}, \tag{1}$$

where $\mathcal{G}(\omega_n, x_1, x_2)$ is the Matsubara Green function of electrons (see Eq. (37.13)) modified appropriately by the zero boundary conditions at $x = 0$. The factor 2 comes from the spin.

Describe how the Friedel oscillations decay with the distance $x$ at a finite temperature $T$. Make sure that your answer at $T \to 0$ reproduces the result of Problem 3 of HW 3.

**2.** [**6 points**] *RKKY in 3D* (See Problem 5 of HW 3)

Calculate at a finite temperature $T$ how the effective interaction $f(\mathbf{r}_1 - \mathbf{r}_2)$ between two impurity spins via exchange with the 3D electron gas depends on the distance $|\mathbf{r}_1 - \mathbf{r}_2|$:

$$\hat{H}_{eff} = f(\mathbf{r}_1 - \mathbf{r}_2)\, \mathbf{S}_1 \cdot \mathbf{S}_2. \tag{2}$$

As in Problem 5 of HW 3,

$$f(\mathbf{r}) \propto T \sum_n \mathcal{G}^2(i\omega_n, \mathbf{r}), \tag{3}$$

where the Matsubara Green function of electrons is given by Eq. (37.13), but you may use the linearized electron dispersion law near $p_F$.

**3.** *Matsubara density-density correlator* [**4 points**]

Matsubara density-density correlation function for a Fermi gas is defined as

$$\Pi(\mathbf{r} - \mathbf{r}', \tau - \tau') = i\langle \mathcal{T}\, \hat{n}(\mathbf{r}, \tau)\, \hat{n}(\mathbf{r}', \tau')\rangle, \tag{4}$$

where $\mathcal{T}$ is the chronological product with respect to the Matsubara time $\tau$.

Derive a general expression for the density-density correlator (4) in the momentum representation, $\Pi(i\Omega_n,\mathbf{q})$. In order to do that, draw the Feynman diagram that corresponds to the calculation of (4) and express $\Pi(i\Omega_n,\mathbf{q})$ in terms Matsubara Green's functions of the noninteracting electrons, $\mathcal{G}(i\omega_m,\mathbf{p}) = 1/[i\omega_m - \epsilon_\mathbf{p} + \mu]$. Summing over the intermediate frequency $\omega_m$ of the loop, obtain the following expression

$$\Pi(i\Omega_n,\mathbf{q}) = 2\int \frac{d^3p}{(2\pi)^3}\,\frac{f(\epsilon_{\mathbf{p}+\mathbf{q}}) - f(\epsilon_\mathbf{p})}{i\Omega_n - \epsilon_{\mathbf{p}+\mathbf{q}} + \epsilon_\mathbf{p}}, \tag{5}$$

where $f(\epsilon)$ is the thermal Fermi distribution function.

Useful formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2 + a^2} = \frac{\pi}{2a}\tanh\frac{\pi a}{2}. \tag{6}$$

(Can you derive this formula?)

**4. [8 points]** *Fluctuations of lattice displacements*

Let us consider the correlator of lattice displacements $\mathbf{u}(\mathbf{r},t)$ at the same time $t$, but at different points in space $\mathbf{r}$:

$$C(\mathbf{r}) = \langle u_\parallel(\mathbf{r})\, u_\parallel(0)\rangle, \tag{7}$$

where $u_\parallel(\mathbf{r}) = \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}\,(\mathbf{k}\cdot\mathbf{u}_\mathbf{k})/|\mathbf{k}|$ is the longitudinal component of displacement.

Express $C(\mathbf{r})$ in terms of the Matsubara Green function of phonons (see Problem 1 of HW 4). Separate the quantum contribution $C_0(\mathbf{r}) = \lim_{T\to 0} C(\mathbf{r})$ and the thermal contribution $C_T(\mathbf{r}) = C(\mathbf{r}) - C_0(\mathbf{r})$. When performing summation over the Matsubara frequencies, use the following formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a}\coth(\pi a). \tag{8}$$

(Can you derive this formula?)

For the acoustic phonons, determine qualitatively how $C_0(\mathbf{r})$ and $C_T(\mathbf{r})$ behave at long distances $r$ in the 1D, 2D, and 3D cases. Do they diverge? What can you say about the long-range order in 1D, 2D, and 3D crystals on the basis of these results?

**5. [4 points]** *Green function at a finite temperature*

The ordinary (not Matsubara) Green function at a finite temperature is defined by Eq. (36.12),

$$iG(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = \langle \mathcal{T}\hat\psi(\mathbf{r}_1,t_1)\,\hat\psi^+(\mathbf{r}_2,t_2)\rangle, \tag{9}$$

where $\mathcal{T}$ represents chronological ordering with respect to the ordinary (not Matsubara) time $t$, and the averaging is done with respect to the Gibbs distribution (36.1).

Using the expansion $\hat\psi(\mathbf{r},t) = \sum_\mathbf{p} e^{i\mathbf{k}\cdot\mathbf{r} - i(\varepsilon_n-\mu)t}\,\hat a_\mathbf{p}$ and the definition (9), calculate $G(\omega,\mathbf{p})$ in the frequency-momentum representation for noninteracting particles. Show that

$$G(\omega,\mathbf{p}) = (1\mp n_\mathbf{p})G^R(\omega,\mathbf{p}) + n_\mathbf{p}G^A(\omega,\mathbf{p}) = \frac{1\mp n_\mathbf{p}}{\omega - \varepsilon_\mathbf{p} + \mu + i0} + \frac{n_\mathbf{p}}{\omega - \varepsilon_\mathbf{p} + \mu - i0}, \tag{10}$$

where the signs $\mp$ refer to fermions and bosons, and $n_\mathbf{p}$ is the thermal distribution function for fermions or bosons.

Check whether Eq. (10) agrees with Eqs. (36.18) and (36.23).