**Homework #7** — Phys625 — Spring 2002
**Deadline: Thursday, April 11, 2002.**
Turn in homework in the class or put it in the box on the door of Phys 2314 by 10 a.m.

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**Do not forget to write your name and the homework number!**
Equation numbers with the period, like (3.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

# Bose Condensation and Superfluidity (Ch. III)

**1.** *Wave function of the condensate (§26, §30)*

Let us consider, as in §30, a slightly non-ideal Bose gas, where almost all of the particles belong to the condensate at zero temperature. Then we can replace the second-quantized particle operator $\hat{\Psi}(\mathbf{r},t)$ (26.1) by the non-operator wave function of the condensate $\Psi(\mathbf{r},t)$ (which is denoted by $\Xi$ in the book). The Hamiltonian of system is obtained by replacing $\hat{\Psi}(\mathbf{r},t) \to \Psi(\mathbf{r},t)$:

$$H = \int d^3r \left( \frac{\hbar^2}{2m} \left| \frac{\partial \Psi}{\partial \mathbf{r}} \right|^2 - \mu |\Psi|^2 \right) + \frac{1}{2} \int d^3r \, d^3r' \, |\Psi(\mathbf{r})|^2 U(\mathbf{r}-\mathbf{r}') |\Psi(\mathbf{r}')|^2, \qquad (1)$$

where $U(\mathbf{r})$ is the interaction potential.

**(a)** [**2 points**] Let us parametrize $\Psi(\mathbf{r},t)$ by its phase and amplitude:

$$\Psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\, e^{i\phi(\mathbf{r},t)}. \qquad (2)$$

Since the Bose gas is only slightly non-ideal, particle density approximately equal to the condensate density: $n \approx n_0$ in Eq. (2).

Substituting (2) into (1), obtain $H$ in terms of $n(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$.

**(b)** [**2 points**] Consider the uniform case first, where all gradients vanish: $n(\mathbf{r},t) = n_0$ and $\phi(\mathbf{r},t) = \text{const}$.

Minimizing $H(n_0)$ with respect to the condensate density $n_0$, determine the chemical potential $\mu$ necessary to produce the given concentration of particles $n \approx n_0$. What is the sign of $\mu$, when $U > 0$ (repulsion)? Compare with the sign of $\mu$ for noninteracting bosons above Bose condensation temperature. What happens if $U < 0$ (attraction)?

**(c)** [**4 points**] Since $n(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$ are canonically conjugated variables (see Eq. (24.7) in §24), obtain the (classical) Hamiltonian equations of motion for $n(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$:

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = \frac{\delta H(n,\phi)}{\delta \phi(\mathbf{r},t)}, \qquad \frac{\partial \phi(\mathbf{r},t)}{\partial t} = -\frac{\delta H(n,\phi)}{\delta n(\mathbf{r},t)} \qquad (3)$$

(i.e. substitute Eq. (1) expressed in terms of $n$ and $\phi$ into Eq. (3)). Interpret the obtained equations physically (see Eq. (26.12)).

**(d)** [**4 points**] Now let us consider small fluctuations near the minimum, i.e. represent $n(\mathbf{r},t) = n_0 + \delta n(\mathbf{r},t)$. Expand Eq. (3) to the first order in $\delta n(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$, and obtain coupled equations of motion for $\delta n(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$. Determine their eigenfrequences $\omega_k$.

**(e)** [**2 points**] Obtain the eigenenergies of excitations $E_k = \hbar\omega_k$ for the following cases

- Short-range interaction: $\tilde{U}(k) = U_0$,
- Coulomb interaction (with a neutralizing uniform background of the opposite electric charge): $\tilde{U}(k) = 4\pi e^2/k^2$,

where $\tilde{U}(k)$ is the interaction potential in the momentum representation. Compare results with Eq. (25.10) and the one after Eq. (33.14) in the textbook.

**2.** *Charged Bose gas*

Consider Bose particles in 3D with the parabolic dispersion law $\varepsilon_k = k^2/2m$, concentration $n$, and electric charge $e$, moving in a neutralizing uniform background of the opposite electric charge.

**(a)** [**6 points**]
Show that the spectrum of excitation has the form

$$E_k \approx \sqrt{\varepsilon_k^2 + (\hbar\Omega_{pl})^2}, \qquad \Omega_{pl}^2 = 4\pi n e^2/m \tag{4}$$

where $\Omega_{pl}$ is the plasma frequency (see HW 2). Obtain Eq. (4) by making the Bogolyubov transformation as in §25 and compare with your result in Problem 1e.

**(b)** [**4 points**]
Calculate the so-called *depletion parameter*, the difference between the total density and the density of the condensate:

$$n - n_0 = \int \frac{d^2k}{(2\pi)^3}\, v_k^2, \tag{5}$$

where $v_k$ is the parameter of the Bogolyubov transformation (see §25, particularly the derivation of Eq. (25.17)). The answer should be expressed as a product of dimensional factors and a dimensionless convergent integral that can be calculated numerically.

**(c)** [**4 points**]
Derive a formula for the ground-state energy $E_0$ of the system, similar to Eq. (25.13), and calculate $E_0$. The answer should be expressed as a product of dimensional factors and a dimensionless convergent integral that can be calculated numerically.

**(d)** [**6 points**]
Calculate Green's functions, as in §33, and determine the excitations eigenenergies from the poles of Green's functions. Compare the result with Problem 2a.

**(e)** **[4 points]**
Show that Green's functions can be represented in the form (compare with Eq. (33.14)):

$$G(\omega,k)=\frac{u_k^2}{\omega-E_k+i0}-\frac{v_k^2}{\omega+E_k-i0},\quad F(\omega,k)=-\frac{u_kv_k}{\omega-E_k+i0}+\frac{u_kv_k}{\omega+E_k-i0}, \tag{6}$$

where $u_k$ and $v_k$ are the coefficients of the Bogolyubov transformation:

$$u_k^2=\frac{\varepsilon_k+n_0\tilde{U}_k}{2E_k}+\frac{1}{2},\quad v_k^2=\frac{\varepsilon_k+n_0\tilde{U}_k}{2E_k}-\frac{1}{2} \tag{7}$$

Using Eq. (6), show that Eq. (31.6) gives the same result for the depletion factor as Eq. (25.17) based on the Bogolyubov transformation.

**3.** Following §23, for the system considered in Problem 2,

**(a)** **[2 points]** Determine the critical velocity.

**(b)** **[2 points]** Crudely estimate the temperature dependence of the the normal density $\rho_n(T)$ at low temperatures.

**4.** *Moving Bose condensate*

Suppose Bose condensation occurs in a state with momentum $q$, so that the wave function of the condensate if $\Psi(bfr,t)=\sqrt{n_0}e^{i\mathbf{q}\cdot\mathbf{r}}$. According to Eq. (26.12), this means that the condensate moves with velocity $\mathbf{v}=\mathbf{q}/m$.

**(a)** **[6 points]** Using either the Bogolyubov transformation as in §25, or determining the self-energies and Green's function as in §33, determine the eigenspectrum $E_\mathbf{k}$ of the excitations. Consider the case of short-range interaction $\tilde{U}(k)=U_0$.

What conservation of momentum tells us about the momenta of particle pairs created from the condensate?

What is $\mu$ in this case? (Do not forget to include the kinetic energy of moving condensate.)

Sketch the spectrum $E_\mathbf{k}$ as a function of k, for $\mathbf{k}$ parallel and antiparallel to $\mathbf{q}$. What happens to the spectrum when the velocity $v$ approaches to the critical velocity?

**(b)** **[4 points]** How does depletion factor depend on the velocity $v$ (for $v$ less than critical)?

Calculate the total momentum of the system and show that it is equal to $nm\mathbf{v}$, not $n_0m\mathbf{v}$, i.e. all particle carry the momentum, not just the condensate.