Electron-phonon interaction (§§12, 13, 14, 64, 65)

1. Green’s function of phonons

(a) [2 points] In Problem 2 of HW 1, we introduced the second-quantized operator of lattice displacements:

$$\hat{u}(r, t) = \sum_{k, \alpha} \mathbf{e}_{k, \alpha} \sqrt{\frac{\hbar}{2V \omega_{0, \alpha}(k)}} \left( \hat{c}_{k, \alpha} e^{i \mathbf{k} \cdot \mathbf{r} - i \omega_{0, \alpha} t} + \hat{c}_{k, \alpha}^+ e^{-i \mathbf{k} \cdot \mathbf{r} + i \omega_{0, \alpha} t} \right).$$  \hspace{1cm} (1)

Here we made a number of generalizations:

- The phonon operators are denoted as \( \hat{c} \) instead of \( \hat{a} \).
- We use the continuous coordinate \( x = na \), where the \( a \) is the inter-site distance, instead of the discrete \( n \). Correspondingly, \( \rho = m/a \) is the mass density of the lattice.
- The crystal is 3D, so the coordinate \( \mathbf{r} \) and momentum \( \mathbf{k} \) are 3D vectors.
- The displacement \( \mathbf{u} \) is also a 3D vector, characterized by three polarizations \( \alpha \) and polarization vectors \( \mathbf{e}_{k, \alpha} \). We will consider only the longitudinal phonons with \( \mathbf{e}_{k,\parallel} || \mathbf{k} \) and ignore the transverse phonons with \( \mathbf{e}_{k,t} \perp \mathbf{k} \).
- The time dependence appears because we consider \( \hat{u} \) in the Heisenberg representation. Correspondingly, \( \omega_{0,\alpha}(\mathbf{k}) \) is the frequency of a phonon with momentum \( \mathbf{k} \) and polarization \( \alpha \).
- The volume \( V \) in denominator provides correct normalization.

Show that the following operator gives the local deviation of the lattice mass density from the equilibrium value:

$$\hat{\rho}'(\mathbf{r}, t) = \rho \text{div} \hat{u}(\mathbf{r}, t) = \rho \frac{\partial \hat{u}}{\partial \mathbf{r}} = \sum_k i \mathbf{k} \sqrt{\frac{\hbar \rho}{2V \omega_{0, \alpha}(k)}} \left( \hat{c}_{k, \alpha} e^{i \mathbf{k} \cdot \mathbf{r} - i \omega_{0, \alpha} t} - \hat{c}_{k, \alpha}^+ e^{-i \mathbf{k} \cdot \mathbf{r} - i \omega_{0, \alpha} t} \right).$$  \hspace{1cm} (2)

Notice that Eq. (2) is consistent with the definition (24.10) for the acoustic phonons with \( \omega_0 = sk \), where \( s \) is the sound velocity (denoted by \( u \) in the book).

(b) [4 points] Green’s function for phonons is defined similarly to that of fermions, with no sign change upon permutation of operators:

$$D(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = -i \langle T \hat{\rho}'(\mathbf{r}_1, t_1) \hat{\rho}'(\mathbf{r}_2, t_2) \rangle = -i \left\{ \begin{array}{ll}
\langle \hat{\rho}'(\mathbf{r}_1, t_1) \hat{\rho}'(\mathbf{r}_2, t_2) \rangle, & t_1 > t_2 \\
\langle \hat{\rho}'(\mathbf{r}_2, t_2) \hat{\rho}'(\mathbf{r}_1, t_1) \rangle, & t_1 < t_2
\end{array} \right\}.$$  \hspace{1cm} (3)

where the averaging is done over the ground state with no phonons present. Using the definition (2), calculate the phonon Green’s function (3).
Electron-phonon interaction is given by Eq. (64.2), which is consistent with the result of Problem 4 of HW in the limit of long-wavelength acoustic phonons. The rules of diagram technique for electron-phonon interaction are described in §64. In the following four problems you are asked to calculate the electron self-energy function $\Sigma(\omega, \mathbf{p})$ in 3D due to interaction with acoustic phonons, which is given by the diagram (64.5) and Eq. (65.2).

2. [8 points] Polaron in weak-coupling limit

Calculate $\Sigma(\omega, \mathbf{p})$ for a single electron interacting with phonons. The single electron (as opposed to electron gas) means that the Fermi energy $\mu \to 0$. Thus use the following Green’s function for the electron line in the diagram: $G(\omega, \mathbf{p}) = (\omega - p^2/2m + i0)^{-1}$. (Notice the absence of sgn$\omega$.)

Consider $\Sigma(\omega, \mathbf{p})$ close the the “mass surface” $\omega = p^2/2m$ and for small momenta $p \ll m_s$. Obtain the expansion:

$$\Sigma(\omega, \mathbf{p}) = \epsilon_0 - \alpha_1 \left( \omega - \frac{p^2}{2m} \right) - \alpha_2 \frac{p^2}{2m}.$$  \hspace{1cm} (4)

Show that $\alpha_2$ determines the renormalization of electron mass $m$, $\alpha_1$ the renormalization of the coefficient $Z$, and $\epsilon_0$ gives the polaron binding energy. Find the effective mass of polaron and comment whether it is heavier or lighter than the bare electron mass.

Hint: When doing the integrals over momentum, use the change of integration variables described in the paragraph after Eq. (65.4), but do not assume that the momenta are close to $p_F$, because there is no Fermi surface in this problem.

3. [8 points] Cherenkov radiation of sound

$\Sigma(\omega, \mathbf{p})$ found in Part 2 has an imaginary part when $v = p/m > s$, because a supersonic electron can emit phonons.

Calculate the decay rate of electron due to emission of phonons, $\gamma = -\text{Im}\Sigma$.

Represent $\text{Im}\Sigma$ as $\int W(\theta) \, d\theta$, where $\theta$ is the angle between the momenta of phonon and electron, and find the angular distribution function $W(\theta)$ for phonon emission.

Calculate the phonon emission rate per unit time directly using Fermi golden rule for electron-phonon interaction (64.2) and compare the results.

4. [6 points] Decay rate for electron gas

Calculate (estimate) the decay rate $-\text{Im}\Sigma(\omega)$ due to interaction with phonons for an electron gas for $\omega$ much smaller than the Fermi energy $\mu$. (In this and the next problem, we consider the case $\mu \neq 0$.)

5. [6 points] Renormalization of electron spectrum

Calculate (estimate) $\text{Re}\Sigma(\omega, \mathbf{p})$ for an electron gas and show that it depends on $\omega$, but practically does not depend on $\mathbf{p}$. Show that $\text{Re}\Sigma(\omega)$ is significant only for $\omega \ll \omega_D$, where $\omega_D = s k_D$ is the Debye energy and $k_D$ is the Debye momentum (the upper momentum cutoff for phonons). Show that the electron mass increases for such energies.