Homework #2 — Phys625 — Spring 2002
Deadline: Thursday, February 14, 2002.
Turn in homework in the class or put it in the box on the door of Phys 2314 by 10 a.m.

Victor Yakovenko, Associate Professor
Office: Physics 2314
Phone: (301)-405-6151
E-mail: yakovenk@physics.umd.edu
Web page: http://www2.physics.umd.edu/~yakovenk/teaching/phys625.spring2002

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

Fermi Liquid Theory

1. *Plasma oscillations*

(a) [**6 points**] Coulomb interaction between electrons gives the following nonlocal-in-space correction to the energy of quasiparticle:

$$\delta\epsilon(\mathbf{r}) = \int \frac{d^3\mathbf{r}'\, d^3\mathbf{p}}{(2\pi)^3} \frac{e^2\, \delta n(\mathbf{p}, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{1}$$

Repeat the derivation of zero sound given in §4 using Eq. (1). Show that Eq. (4.15) acquires the following form:

$$\frac{\omega}{2v_F k} \ln\left(\frac{\omega + v_F k}{\omega - v_F k}\right) - 1 = \frac{k^2}{4\pi e^2 \nu_F}, \tag{2}$$

where ν_F is the density of states given by Eq. (2.5).

From Eq. (2), determine the frequency ω_0 of the collective mode in the limit $k \to 0$. It is called the plasma frequency.

(b) [**2 points**] Let us derive the frequency of plasma oscillations in a different way.

Consider the continuity equation for the electron gas:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) \approx \frac{\partial\, \delta n}{\partial t} + n_0 \nabla \cdot \mathbf{v} = 0, \tag{3}$$

where $n = n_0 + \delta n$, and $\mathbf{v}$ is the velocity of hydrodynamic flow. Let us differentiate Eq. (3) in time:

$$\frac{\partial^2 \delta n}{\partial t^2} + n_0 \nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = 0. \tag{4}$$

Using Newton's equation of motion $\partial\mathbf{v}/\partial t = -e\mathcal{E}/m$ and Poisson's equation $\nabla \cdot \mathcal{E} = -4\pi e\, \delta n$, where $\mathcal{E}$ is electric field, transform Eq. (4) to the following form

$$\frac{\partial^2 \delta n}{\partial t^2} = -\omega_0^2\, \delta n \tag{5}$$

and determine the frequency ω_0. Eq. (5) describes local oscillations of electron density.

2. **[8 points]** *Fermi-liquid modes in 2D*

Let us consider a Fermi liquid in two-dimensional (2D) space, where the Fermi sphere is a circle. The deviation of the quasiparticle distribution function $\nu(\theta)$ is a function of the angle θ on the circle. Suppose the Landau interaction function is a constant: $F(\mathbf{n}-\mathbf{n}') = F_0$. Using the Fourier expansion $\nu(\theta) = \sum_m \nu_m \exp(im\theta)$, show that the kinetic equation becomes

$$\omega\nu_m = \frac{1}{2}kv_F(\tilde{\nu}_{m+1} + \tilde{\nu}_{m-1}), \quad \text{where} \quad \tilde{\nu}_m = \begin{cases} \nu_m, & \text{when } m \neq 0, \\ (1+F_0)\nu_0, & \text{when } m = 0. \end{cases} \tag{6}$$

Because Eq. (6) is invariant with respect to translations $m \to (m\pm1)$, except at $m=0$, we can look for its solutions in the form $\nu_m = A^{\pm}\exp(i\alpha m)$ for $m>0$ and $m<0$, and then match them at $m=0$.

Show that for any real $0<\alpha<\pi$, there exist two such solutions describing quasiparticles with $\omega = v_F k\cos\alpha$ and $\theta = \pm\alpha$. [By parity, the two solutions can be written as $\nu_m = \sin(\alpha m)$ and as $\nu_m = \cos(\alpha m \pm \lambda)$ for $m>0$ and $m<0$, and the parameter λ needs to be determined.]

Besides, for $F_0 > 0$, there exist one solution with a complex α, which decays exponentially at $m \to \pm\infty$. It corresponds to the zero-sound collective mode. Find its velocity ω/k and angular dependence $\nu(\theta)$. Show that the frequency of this solution lies above the quasiparticle continuum $\omega(k) > v_F k$.

[Notice a mathematical analogy between this problem and the problem of a 1D quantum particle subject to the $\delta(x)$ potential.]

3. **[6 points]** *Electron-hole continuum*

Let us introduce the operator $\hat{c}^{+}_{\mathbf{p},\mathbf{k}}$ that creates a hole with momentum $\mathbf{p}$ and an electron with momentum $\mathbf{p}+\mathbf{k}$ out of the Fermi sea:

$$\hat{c}^{+}_{\mathbf{p},\mathbf{k}} = \hat{a}^{+}_{\mathbf{p}+\mathbf{k}}\hat{a}_{\mathbf{p}}, \quad \text{where} \quad |\mathbf{p}| < p_F \quad \text{and} \quad |\mathbf{p}+\mathbf{k}| > p_F. \tag{7}$$

The energy of such an excitation is

$$E_{\mathbf{p},\mathbf{k}} = \epsilon_{\mathbf{p}+\mathbf{k}} - \epsilon_{\mathbf{p}} = \frac{|\mathbf{p}+\mathbf{k}|^2 - |\mathbf{p}|^2}{2m}. \tag{8}$$

With the restriction (7) on $\mathbf{p}$ and $\mathbf{k}$, outline the area covered by these excitation on a plot of $E(k)$ for all permitted values of p. This is the so-called electron-hole continuum of excitations. On the same plot, also sketch the dispersions $\omega(k)$ of the zero-sound and plasma-oscillations collective modes.