

Let's find the period of oscillation for an anharmonic oscillator. Consider a particle of mass  $m$  moving in a potential

$$V(x) = \frac{1}{2}kx^2 + ax^4.$$

This is close to a harmonic oscillator when the amplitude  $x_m$  is not too large. More precisely, the condition is  $ax_m^2/k \ll 1$ , where  $\ll$  means "much less than".

1. Show that the period of an oscillation with total energy  $E$  and amplitude  $x_m$  is given by

$$T = \sqrt{8m} \int_0^{x_m} \frac{dx}{\sqrt{E - V}}.$$

(*Hint:* The time  $dt$  it takes the particle to travel a distance  $dx$  is  $dx/(dx/dt)$ .)

2. Evaluate the integral for the case  $a = 0$  and show that you recover the usual result for a harmonic oscillator. For good practice, before carrying out the integral adopt units adapted to the problem at hand, in which  $m = 1$  and  $k = 1$  and  $x_m = 1$ . (How do you know such units exist?) This will simplify the intermediate steps. After obtaining a result, express your answer in arbitrary units by using dimensional analysis to insert the unique combination of  $m$ ,  $k$ , and  $x_m$  with the units of time.
3. Now do the anharmonic case, but don't do it exactly. After all, the  $ax^4$  term is almost always followed by an infinite number of terms in a series expansion, so the exact result for the above potential would not be physically the right answer anyway. Instead, expand<sup>1</sup> and keep only the terms of zeroth and first order in the small number  $ax_m^2/k$ . Again use our special units, and only at the end use dimensional analysis to put back in the right combinations of the constants.
4. Apply your result to the simple pendulum. How does the period depend on the amplitude  $\theta_m$  to lowest order? How large must  $\theta_m$  be if the anharmonic correction is 1% of the harmonic period? How about 10%?

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<sup>1</sup>Use the Taylor expansion  $(1 + \epsilon)^{-1/2} = 1 - \frac{1}{2}\epsilon + O(\epsilon^2)$ .