Homework #4 — Phys625 — Spring 2004
Deadline: Wednesday, May 5, 2004.
Turn in homework in the class or put it in the box on the door of Phys 2314 by 3 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the periods, like (3.2.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

Generalized Susceptibilities at a Finite Temperature

1. *Matsubara density-density correlator* [**4 points**]

Matsubara density-density correlation function for a Fermi gas is defined as

$$\Pi(\mathbf{r}-\mathbf{r}',\tau-\tau') = i\langle\mathcal{T}\,\hat{n}(\mathbf{r},\tau)\,\hat{n}(\mathbf{r}',\tau')\rangle, \tag{1}$$

where $\mathcal{T}$ is the chronological product with respect to the Matsubara time τ.

Derive a general expression for the density-density correlator (1) in the momentum representation, $\Pi(i\Omega_n,\mathbf{q})$. In order to do that, draw the Feynman diagram that corresponds to the calculation of (1) and express $\Pi(i\Omega_n,\mathbf{q})$ in terms Matsubara Green's functions of the noninteracting electrons, $\mathcal{G}(i\omega_m,\mathbf{p}) = 1/[i\omega_m - \epsilon_{\mathbf{p}} + \mu]$. Summing over the intermediate frequency $\omega_m = (2m+1)\pi T$ of the loop, obtain the following expression

$$\Pi(i\Omega_n,\mathbf{q}) = 2\int\frac{d^3p}{(2\pi)^3}\,\frac{f(\epsilon_{\mathbf{p}+\mathbf{q}}) - f(\epsilon_{\mathbf{p}})}{i\Omega_n - \epsilon_{\mathbf{p}+\mathbf{q}} + \epsilon_{\mathbf{p}}}, \tag{2}$$

where $f(\epsilon)$ is the thermal Fermi distribution function.

Useful formula:

$$\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2 + a^2} = \frac{\pi}{2a}\tanh\frac{\pi a}{2}. \tag{3}$$

(Can you derive this formula?)

2. *From Matsubara to real frequencies* [**4 points**]

The (retarded) dynamical susceptibility $\chi(\omega)$ is given by the Kubo formula [see Sec. 6.4]:

$$\chi(\omega) = i\int_0^\infty dt\, e^{i\omega t}\langle[\hat{A}(t),\hat{B}(t)]\rangle_T, \tag{4}$$

where t and ω are the real time and frequency, $\hat{A}(t)$ and $\hat{B}(t)$ are the operators in the Heisenberg representation, and the averaging is performed over the Gibbs distribution with the temperature T.

Consider the case where the operators $\hat{A}$ and $\hat{B}$ are bilinear in creation and destruction operators:

$$\hat{A}(t) = \sum_{kl} A_{kl}e^{-i(E_k-E_l)t}\hat{a}_k^+\hat{a}_l, \qquad \hat{B}(t) = \sum_{kl} B_{kl}e^{-i(E_k-E_l)t}\hat{a}_k^+\hat{a}_l. \tag{5}$$

where A_{kl} and B_{kl} are the matrix elements of $\hat{A}$ and $\hat{B}$ between the energy eigenstates k and l.

Substituting Eq. (5) into Eq. (4) and performing the thermal averaging for non-interacting particles, show that

$$\chi(\omega) = \sum_{kl} A_{kl} B_{lk} \frac{f(E_l) - f(E_m)}{\omega - E_l + E_k + i0}, \tag{6}$$

where $f(E)$ is the thermal Fermi or Bose distribution function.

Notice the similarity between Eq. (6) and Eq. (2), provided the indices k and l represent the momenta $\mathbf{p}$ and $\mathbf{q}$. It is clear that Eq. (6) can be obtained from Eq. (2) by the analytical continuation $i\Omega_n \to \omega + i0$.

3. *Transition temperature of the Peierls instability*

Let us consider an external potential $U(\mathbf{r})$ acting as a perturbation on electrons:

$$\hat{H}_1 = \int d^3r\, U(\mathbf{r})\hat{n}(\mathbf{r}). \tag{7}$$

Then, the density response function is

$$n(\mathbf{q}) = \chi(\mathbf{q})\, U(\mathbf{q}) \tag{8}$$

Here we are interested in the static case $\omega = 0$.

Let us consider 1D electron gas with the Fermi momentum k_F. In the following calculations, use the linearized dispersion law for electrons:

$$\begin{aligned} \epsilon_k - \mu &\approx v_F\, \delta k \qquad \text{where } \delta k = k - k_F \text{ for } k \text{ close to } +k_F \\ \epsilon_k - \mu &\approx -v_F\, \delta k \qquad \text{where } \delta k = k + k_F \text{ for } k \text{ close to } -k_F, \end{aligned}$$

where v_F is the Fermi velocity.

(a) [**4 points**] Using Eq. (2) or (6), calculate the response function χ in Eq. (8) $q = 2k_F$ at a temperature T for noninteracting electrons.

(b) [**4 points**] Suppose electrons interact with an amplitude g. Considering the following series of diagrams:

where the circle represents the interaction vertex g, calculate the renormalized susceptibility χ at $q = 2k_F$. Determine the temperature T_c where χ diverges. Interpret the result.

Hint: The calculations are similar to those for the superconducting instability considered in Sec. 10.2.